

Hölder stability for parametric vector equilibrium problems

Le Xuan Dai^{1,2}, Tran Quoc Duy³, Tran Ngoc Tam^{4,*}



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¹Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268, Ly Thuong Kiet, Dien Hong Ward, Ho Chi Minh City

²Vietnam National University, Ho Chi Minh City, Vietnam (VNU-HCM)

³Department of Mathematics, FPT University, Can Tho, Vietnam.

⁴Faculty of Mathematics, College of Natural Sciences, Can Tho University, Can Tho, Vietnam

Correspondence

Tran Ngoc Tam, Faculty of Mathematics, College of Natural Sciences, Can Tho University, Can Tho, Vietnam

Email: tntam@ctu.edu.vn

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ABSTRACT

It is known that stability is a fundamental and important topic in optimization theory. The main objective of this field is to study of behavior of solutions to parametric problems when the data of the problems undergoes small changes. Therefore, stability for optimization-related problems holds both theoretical and practical significance, which has been used as a useful tool in post-optimal analysis. As usual, the stability can be understood in two ways: the first involves the upper and lower semicontinuity and the continuity of solution maps to parametric problems, referred to as qualitative stability; the second encompasses the Hölder/Lipschitz continuity of solution maps of the problems, termed quantitative stability. For parametric problems with the parameters perturbed in spaces of parameters, several researchers have established various types of stability conditions, drawing on the continuity properties of objective maps and constrained maps, along with their convexity and monotonicity properties. In this paper, we are concerned with parametric vector equilibrium problems in normed spaces and aim at to improve some recent results. To achieve this, we first introduce a relaxed concept of cone-quasiconcavity, specifically the cone-lower level quasiconcavity. Also, necessary and sufficient conditions are investigated. We provide an example to illustrate that this new condition is strictly weaker than the cone-quasiconcavity. After that, we employ this new condition together with Hölder properties of the constrained map and the objective map to establish sufficient conditions for the stability in the sense of Hölder continuity of solution maps for the reference problems via the linear scalarization method. A special case of the reference problems is also discussed. Our results in this paper represent improvements over the existing ones in the literature.

Key words: Vector equilibrium problem, Hölder continuity, parametric problem, approximate solution

INTRODUCTION

The so-called equilibrium problem, in the sense of^{17,20,27}, is a common formulation for describing various problems arising in different fields of mathematics. It encompasses several important problems such as the optimization problem, the variational inequality problem, the fixed-point and coincidence problems, Nash equilibrium problem, and more, see e.g.,^{16,22}. There are a lot of works devoted to existence conditions for this problem and its extensions, see e.g.,^{1-3, 6, 7, 18, 21, 28} and the references therein. Stability analysis of solutions, including (semi)continuity in the sense of Berge and Hausdorff, as well as in the sense of Hölder/Lipschitz, also serves as a motivation for many researchers, see e.g.,^{4, 5, 8, 10-12, 14, 15, 19, 26, 29, 30}.

Observe that most of the previous results on the Hölder/Lipschitz continuity of the solution maps impose assumptions related to strong monotonicity/convexity conditions. As a consequence, the solution set of the problem is a singleton in the neighborhood of the reference point. It is known that the solution map is set-valued in general. Therefore, many attempts have been made to overcome this drawback. In^{23, 24}, assumptions involving to the solution set are imposed on the data; however, these assumptions are difficult to verify due to the unknown solution set. In^{9, 13}, verifiable conditions are utilized, and the Hölder continuity for approximate solution maps are obtained in the case of non-unique solutions.

The aim of this paper is to investigate the Hölder continuity of the approximate solution map to a parametric vector equilibrium problem, which improves upon the results presented in^{9, 13, 25}. To be more precise, we demonstrate that the conditions established in^{9, 13, 25} can be relaxed to the weaker ones. Additionally, we provide comprehensive proofs of the main results presented in the aforementioned papers.

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The remainder of this paper is organized as follows. Section 2, we state the parametric vector equilibrium problem and recall some concepts needed in what follows. Section 3 is dedicated to establishing sufficient conditions for the Hölder continuity of the approximate solution map under milder assumptions.

PRELIMINARIES

Throughout this paper, we use $\|\cdot\|$ for the norm in any normed space. $d(x, A)$ is the distance from x to subset A . For a normed space X , X^* is the topological dual space of X and $\langle \cdot, \cdot \rangle$ is the duality pairing. $\mathbb{R}_+$ is the set of nonnegative real numbers. $\mathbb{B}(x, r)$ denotes the closed ball of radius $r \geq 0$ centered at x . $\text{int } A$ stands for the interior of A . The diameter of A is

$$\text{diam} A := \sup_{x, x' \in A} \|x - x'\|.$$

For $A, B \subset X$, the Hausdorff distance between A and B is defined by

$$\mathcal{H}(A, B) = \max\{\mathcal{H}^*(A, B), \mathcal{H}^*(B, A)\},$$

where

$$\mathcal{H}^*(A, B) = \sup_{a \in A} d(a, B),$$

and

$$d(x, A) = \inf_{y \in A} d(x, y).$$

From now on, unless otherwise stated, let X, Y, Λ , and M be normed spaces. $A \subset X$ is a nonempty subset, $C \subset Y$ is a pointed closed convex cone with nonempty interior ($\text{int} C \neq \emptyset$). The set-valued map $K: \Lambda \rightrightarrows A$ has nonempty bounded convex values and $f: A \times A \times M \rightarrow Y$ is a vector-valued map. For each $(\lambda, \mu) \in \Lambda \times M$, the parametric vector equilibrium problem under our consideration is as follows:

(WEP): Find $\bar{x} \in K(\lambda)$ such that

$$f(\bar{x}, y, \mu) \notin -\text{int } C, \forall y \in K(\lambda).$$

For every $(\varepsilon, \lambda, \mu) \in \mathbb{R}_+ \times \Lambda \times M$ and $e \in \text{int} C$, we denote the ε -approximate solution set of (WEP) at $(\varepsilon, \lambda, \mu)$ by

$$S(\varepsilon, \lambda, \mu) := \{x \in K(\lambda) : f(x, y, \mu) + \varepsilon e \notin -\text{int} C, \forall y \in K(\lambda)\}.$$

We denote the dual cone of C by

$$C^* := \{\xi \in Y^* : \langle \xi, y \rangle \geq 0, \forall y \in C\}.$$

Then, the set $B_e^* := \{\xi \in C^* : \langle \xi, e \rangle = 1\}$ is a weak* compact base of C^* . For every $\xi \in B_e^*$, the ξ -approximate solution set of (WEP) is denoted by

$$S_\xi(\varepsilon, \lambda, \mu) := \{x \in K(\lambda) : \langle \xi, f(x, y, \mu) \rangle + \varepsilon \geq 0, \forall y \in K(\lambda)\}.$$

Now, we recall some notions needed in what follows.

Definition 2.1.

(a) A map $f: X \rightarrow Y$ is said to be C - l, α -Hölder continuous with respect to (wrt, shortly) e at $x_0 \in X$, if there is a neighborhood V of x_0 such that, for all $x_1, x_2 \in V$,

$$f(x_1) - f(x_2) + l\|x_1 - x_2\|^\alpha e \in C.$$

(b) A map $f: X \times X \rightarrow Y$ is said to be C -($l_1, \alpha_1, l_2, \alpha_2$)-Hölder continuous wrt e at (x_0, y_0) if there exists a neighborhood $N_1 \times N_2$ of (x_0, y_0) such that for all $(x_1, y_1), (x_2, y_2) \in N_1 \times N_2$,

$$f(x_1, y_1) - f(x_2, y_2) + (l_1 \|x_1 - x_2\|^{\alpha_1} + l_2 \|y_1 - y_2\|^{\alpha_2}) e \in C.$$

(c) A map $f: X \times X \times X \rightarrow Y$ is said to be $C - (l_1, \alpha_1, l_2, \alpha_2, l_3, \alpha_3)$ -Hölder continuous wrt e at (x_0, y_0, z_0) if there exists a neighborhood $N_1 \times N_2 \times N_3$ of (x_0, y_0, z_0) such that, for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in N_1 \times N_2 \times N_3$,

$$f(x_1, y_1, z_1) - f(x_2, y_2, z_2) + [l_1 \|x_1 - x_2\|^{\alpha_1} + l_2 \|y_1 - y_2\|^{\alpha_2} + l_3 \|z_1 - z_2\|^{\alpha_3}] e \in C.$$

(d) A set-valued map $K: \Lambda \rightrightarrows A$ is said to be l, α -Hölder continuous at $\lambda_0 \in \Lambda$ if there is a neighborhood U of λ_0 such that, for all $\lambda_1, \lambda_2 \in U$,

$$K(\lambda_1) \subset K(\lambda_2) + l \|\lambda_1 - \lambda_2\|^\alpha \mathbb{B}(0, 1).$$

Definition 2.2. Let $f: X \rightarrow Y$ be a vector-valued map and Ω be a convex subset of X . It is said that

(a) f is C -concave on Ω if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$,

$$f(tx_1 + (1 - t)x_2) \in tf(x_1) + (1 - t)f(x_2) + C.$$

(b) f is C -quasiconcave on Ω if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$, either

$$f(tx_1 + (1 - t)x_2) \in f(x_1) + C,$$

or

$$f(tx_1 + (1 - t)x_2) \in f(x_2) + C.$$

(c) f is C -convexlike on Ω (Ω is not necessarily convex) if for any $x_1, x_2 \in \Omega$ and $t \in [0, 1]$, there exists $x_3 \in \Omega$ such that $tf(x_1) + (1 - t)f(x_2) \in f(x_3) + C$.

We now propose a generalized concept related to the quasiconcavity.

Definition 2.3. The map $f: X \rightarrow Y$ is said to be C -lower level quasiconcave on Ω if for any $x_1, x_2 \in \Omega, t \in [0, 1]$ with $f(tx_1 + (1 - t)x_2) \in -\text{int}C$, either $f(x_1) \in -\text{int}C$ or $f(x_2) \in -\text{int}C$.

It is easy to see that f is C -quasiconcave on Ω , then it is C -lower level quasiconcave on Ω . We now provide an equivalent condition to the C -lower level quasiconcavity.

Lemma 2.4. The map $f: X \rightarrow Y$ is C -lower level quasiconcave on Ω if and only if the set $L := \{x \in \Omega: f(x) \in Y \setminus -\text{int}C\}$ is convex.

Proof.

[$\Rightarrow$] Let $x_1, x_2 \in L$, then $f(x_1) \in Y \setminus -\text{int}C$ and $f(x_2) \in Y \setminus -\text{int}C$. Suppose that $f(tx_1 + (1 - t)x_2) \notin Y \setminus -\text{int}C$ with $t \in [0, 1]$, i.e., $f(tx_1 + (1 - t)x_2) \in -\text{int}C$, implying that either $f(x_1) \in -\text{int}C$ or $f(x_2) \in -\text{int}C$. This is a contradiction. Therefore, $f(tx_1 + (1 - t)x_2) \in Y \setminus -\text{int}C$, and hence $tx_1 + (1 - t)x_2 \in L$, which yields that L is convex.

[$\Leftarrow$] Suppose that f is C -lower level quasiconcave on Ω . Suppose that there are $x_1, x_2 \in \Omega, t \in [0, 1]$ with $f(tx_1 + (1 - t)x_2) \in -\text{int}C$ but $f(x_1), f(x_2) \in Y \setminus -\text{int}C$, i.e., $x_1, x_2 \in L$. Hence, $tx_1 + (1 - t)x_2 \in L$, which yields that $f(tx_1 + (1 - t)x_2) \in Y \setminus -\text{int}C$. This is impossible. $\square$

The following example shows that the C -lower level quasiconcavity is strictly weaker than the C -quasiconcavity.

Example 2.5. Let $X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}_+^2, \Omega = [-1, 1]$ and $f(x) = (x^2, x^2)$. It is easy to see that $\{x \in \Omega: (x^2, x^2) \in Y \setminus -\text{int}C\} = [-1, 1]$ is convex, implying that f is C -lower level quasiconcave on Ω . However, f is not C -quasiconcave on Ω . To see this, take $x_1 = -1, x_2 = 1$

and $t = \frac{1}{2}$, one has $f(x_1) = f(x_2) = (1, 1)$ and $f(tx_1 + (1-t)x_2) = f(0) = (0, 0)$. Obviously, $f(x_t) \notin f(x_1) + \mathbb{R}_+^2$ and $f(x_t) \notin f(x_2) + \mathbb{R}_+^2$.

Lemma 2.6. For each $\xi \in B_e^*$, $y \in K(\Lambda)$ and $\mu \in M$, if the map $(x, \varepsilon) \mapsto f(x, y, \mu) + \varepsilon e$ is C -lower level quasiconcave on $A := K(\Lambda) \times (0, +\infty)$, then the following set is convex:

$$\mathcal{L} := \{(x, \varepsilon) \in A : \langle \xi, f(x, y, \mu) \rangle + \varepsilon \geq 0\}.$$

Proof. Take any $(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in \mathcal{L}$, $t \in [0, 1]$ and let $(x_t, \varepsilon_t) := (tx_1 + (1-t)x_2, t\varepsilon_1 + (1-t)\varepsilon_2)$. Suppose that for each $\xi \in B_e^*$,

$$\langle \xi, f(x_t, y, \mu) \rangle + \varepsilon_t < 0$$

then one has $f(x_t, y, \mu) + \varepsilon_t e \in -\text{int} C$. Therefore, either $f(x_1, y, \mu) + \varepsilon_1 e \in -\text{int} C$ or $f(x_2, y, \mu) + \varepsilon_2 e \in -\text{int} C$, which yields that either $\langle \xi, f(x_1) \rangle + \varepsilon_1 < 0$ or $\langle \xi, f(x_2) \rangle + \varepsilon_2 < 0$, a contradiction as $(x_1, \varepsilon_1), (x_2, \varepsilon_2) \in \mathcal{L}$. $\square$

Lemma 2.7. (See [9]) For each $\lambda \in \Lambda$, $x \in K(\lambda)$ and $\mu \in M$, if the mapping $y \mapsto f(x, y, \mu)$ is C -convexlike on $K(\lambda)$, then

$$S(\varepsilon, \lambda, \mu) = \bigcup_{\xi \in B_e^*} S_\xi(\varepsilon, \lambda, \mu).$$

HÖLDER CONTINUITY OF APPROXIMATE SOLUTION MAPS

In this section, we begin with establishing adequate conditions for ensuring the Hölder continuity of the ξ -approximate solution maps. Applying this result with suitable techniques, sufficient conditions for the Hölder property of the approximate solution maps are established.

Theorem 3.1. For (WEP), assume that for each $\xi \in B_e^*$, the ξ -approximate solutions exist in a neighborhood of the considered point $(\lambda_0, \mu_0) \in \Lambda \times M$. Assume further that

- (i) K is m, α -Hölder continuous on Λ ;
- (ii) for each $y \in K(\Lambda)$ and $\mu \in M$, $(x, \varepsilon) \mapsto f(x, y, \mu) + \varepsilon e$ is C -lower level quasiconcave on $K(\Lambda) \times (0, +\infty)$.
- (iii) f is $(l_1, \alpha_1, l_2, \alpha_2, h, \beta)$ -Hölder continuous on $K(\Lambda) \times K(\Lambda) \times M$.

Then, for any $\bar{\varepsilon} > 0$ and $\bar{\xi} \in B_e^*$, there exist open neighborhoods $N(\bar{\xi})$ of $\bar{\xi}$, $N_{\bar{\xi}}(\lambda_0)$ of λ_0 and $N_{\bar{\xi}}(\mu_0)$ of μ_0 such that the mapping $S_{\bar{\xi}}$ is Hölder continuous on $[\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$, i.e.,

$$\mathcal{H}(S_{\bar{\xi}}(\varepsilon_1, \lambda_1, \mu_1), S_{\bar{\xi}}(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1 |\varepsilon_1 - \varepsilon_2| + k_2 \|\mu_1 - \mu_2\|^\beta + k_3 \|\lambda_1 - \lambda_2\|^{\alpha\delta},$$

where $\xi \in N(\bar{\xi})$, $(\varepsilon_i, \lambda_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$, $i = 1, 2$ and k_1, k_2, k_3 are positive and depend on $\bar{\varepsilon}, m, \alpha, h, \beta$, etc.

Proof. Let $\bar{\xi} \in B_e^*$ and $N(\bar{\xi}) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0) \subset B_e^* \times U \times V$ be open, where U and V are neighborhoods of λ_0 and μ_0 , respectively. Let $(\varepsilon_1, \lambda_1, \mu_1), (\varepsilon_2, \lambda_2, \mu_2) \in [\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$ be arbitrarily given. Without loss of generality, we always assume that $\varepsilon_1 < \varepsilon_2$. For the sake of convenience, the proof is divided into three steps.

Step 1. We estimate for $\mathcal{H}(S_{\bar{\xi}}(\varepsilon_1, \lambda_1, \mu_1), S_{\bar{\xi}}(\varepsilon_2, \lambda_1, \mu_1))$ where $0 < \varepsilon_1 < \varepsilon_2$ and $\xi \in N(\bar{\xi})$. Let

$$\rho := \text{diam} K(\lambda_0) + 2m(\text{diam}(U))^\alpha,$$

we first show that for all $(\lambda, \mu) \in U \times V$ and $\xi \in B_e^*$,

$$\mathcal{H}\left(S_{\xi}(\varepsilon_1, \lambda, \mu), S_{\xi}(\varepsilon_2, \lambda, \mu)\right) \leq \rho \varepsilon_2^{-1} |\varepsilon_1 - \varepsilon_2|.$$

Obviously, $S_{\xi}(\varepsilon_1, \lambda, \mu) \subset S_{\xi}(\varepsilon_2, \lambda, \mu)$. Therefore,

$$\mathcal{H}^*\left(S_{\xi}(\varepsilon_1, \lambda, \mu), S_{\xi}(\varepsilon_2, \lambda, \mu)\right) = 0.$$

Take any $x_2 \in S_{\xi}(\varepsilon_2, \lambda, \mu)$ and $x_0 \in S_{\xi}(0, \lambda, \mu)$, by the convexity of $K(\lambda)$, we have that

$$\varepsilon_1 \varepsilon_2^{-1} x_2 + (\varepsilon_2 - \varepsilon_1) \varepsilon_2^{-1} x_0 \in K(\lambda).$$

For all $y \in K(\lambda)$, we have

$$\langle \xi, f(x_0, y, \mu) \rangle \geq 0 \text{ and } \langle \xi, f(x_2, y, \mu) + \varepsilon_2 \rangle \geq 0,$$

i.e.,

$$(x_0, 0), (x_2, \varepsilon_2) \in \mathcal{L},$$

where

$$\mathcal{L} := \{(x, \varepsilon) \in K(\lambda) \times (0, +\infty) : \langle \xi, f(x, y, \mu) \rangle + \varepsilon \geq 0, \forall y \in K(\lambda)\}.$$

By Lemma 2.6, one gets

$$(\varepsilon_1 \varepsilon_2^{-1} x_2 + (\varepsilon_2 - \varepsilon_1) \varepsilon_2^{-1} x_0, \varepsilon_1 \varepsilon_2^{-1} \varepsilon_2 + (\varepsilon_2 - \varepsilon_1) \varepsilon_2^{-1} 0) \in \mathcal{L},$$

which implies

$$\langle \xi, f(\varepsilon_1 \varepsilon_2^{-1} x_2 + (\varepsilon_2 - \varepsilon_1) \varepsilon_2^{-1} x_0, y, \mu) \rangle + \varepsilon_1 \geq 0.$$

Thus, we get $x_1 := \varepsilon_1 \varepsilon_2^{-1} x_2 + (\varepsilon_2 - \varepsilon_1) \varepsilon_2^{-1} x_0 \in S_{\xi}(\varepsilon_1, \lambda, \mu)$. Then,

$$\|x_2 - x_1\| = \varepsilon_2^{-1} |\varepsilon_1 - \varepsilon_2| \|x_2 - x_0\|.$$

We see that $K(\lambda) \subset K(\lambda_0) + mB(0, \|\lambda - \lambda_0\|^\alpha)$ for all $\lambda \in U$ and hence $\text{diam } K(\lambda) \leq \rho$. Therefore, $\|x_2 - x_1\| \leq \rho \varepsilon_2^{-1} |\varepsilon_1 - \varepsilon_2|$, and we get the following estimate:

$$\mathcal{H}^*\left(S_{\xi}(\varepsilon_2, \lambda, \mu), S_{\xi}(\varepsilon_1, \lambda, \mu)\right) \leq \rho \varepsilon_2^{-1} |\varepsilon_1 - \varepsilon_2|.$$

Consequently,

$$\mathcal{H}\left(S_{\xi}(\varepsilon_1, \lambda, \mu), S_{\xi}(\varepsilon_2, \lambda, \mu)\right) \leq \rho \varepsilon_2^{-1} |\varepsilon_1 - \varepsilon_2|.$$

This together with the fact $\varepsilon_2 \in [\bar{\varepsilon}, +\infty)$ implies that

$$\begin{aligned} \mathcal{H}\left(S_{\xi}(\varepsilon_1, \lambda_1, \mu_1), S_{\xi}(\varepsilon_2, \lambda_1, \mu_1)\right) \\ \leq \rho(\bar{\varepsilon})^{-1} |\varepsilon_1 - \varepsilon_2| = k_1 |\varepsilon_1 - \varepsilon_2|, \end{aligned}$$

where $k_1 = \rho(\bar{\varepsilon})^{-1}$.

Step 2. We estimate for $\mathcal{H}\left(S_{\xi}(\varepsilon_2, \lambda_1, \mu_1), S_{\xi}(\varepsilon_2, \lambda_1, \mu_2)\right)$ where $\bar{\varepsilon} < \varepsilon_2$, $\lambda_1 \in N_{\bar{\xi}}(\lambda_0)$, $\xi \in N(\bar{\xi})$ and $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$ with $\mu_1 \neq \mu_2$. First, for each $x, y \in K(U)$ and $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$, by (iii), we have

$$f(x, y, \mu_1) - f(x, y, \mu_2) + h\|\mu_1 - \mu_2\|^\beta e \in C.$$

Then, for each $\xi \in N(\bar{\xi})$,

$$|\langle \xi, f(x, y, \mu_1) \rangle - \langle \xi, f(x, y, \mu_2) \rangle| \leq h \|\mu_1 - \mu_2\|^\beta.$$

Now, we divide Step 2 into two cases.

Case 1: $h \|\mu_1 - \mu_2\|^\beta \leq \varepsilon_2$. Let $r = h \|\mu_1 - \mu_2\|^\beta$, then $0 < r \leq \varepsilon_2$. We claim that

$$S_\xi(\varepsilon_2 - r, \lambda_1, \mu_1) \subset S_\xi(\varepsilon_2, \lambda_1, \mu_2).$$

Indeed, let $\bar{x} \in S_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)$. Then, for all $y \in K(\lambda_1)$,

$$\langle \xi, f(\bar{x}, y, \mu_2) \rangle + \langle \xi, f(\bar{x}, y, \mu_1) \rangle - \langle \xi, f(\bar{x}, y, \mu_2) \rangle + \varepsilon_2 - r \geq 0.$$

Consequently, $\langle \xi, f(\bar{x}, y, \mu_2) \rangle + \varepsilon_2 \geq 0$ for all $y \in K(\lambda_1)$, i.e., $\bar{x} \in S_\xi(\varepsilon_2, \lambda_1, \mu_2)$. Hence, using the result of Step 1, we arrive at:

$$\begin{aligned} & \mathcal{H}^*(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_2)) \\ & \leq \mathcal{H}^*(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)) \\ & \leq \mathcal{H}^*(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)) \end{aligned}$$

As a result,

$$\begin{aligned} & \mathcal{H}^*(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_2)) \leq \rho r \varepsilon_2^{-1} \\ & \leq \rho h (\bar{\varepsilon})^{-1} \|\mu_1 - \mu_2\|^\beta. \end{aligned}$$

With the same arguments, we get

$$\mathcal{H}^*(S_\xi(\varepsilon_2, \lambda_1, \mu_2), S_\xi(\varepsilon_2, \lambda_1, \mu_1)) \leq \rho h (\bar{\varepsilon})^{-1} \|\mu_1 - \mu_2\|^\beta.$$

Therefore,

$$\mathcal{H}(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_2)) \leq \rho h (\bar{\varepsilon})^{-1} \|\mu_1 - \mu_2\|^\beta.$$

Case 2: $h \|\mu_1 - \mu_2\|^\beta > \varepsilon_2$. Let $\rho_1 = \text{diam } V$. Then, there exists a natural number n such that $\frac{\rho_1}{n} \leq \left(\frac{\bar{\varepsilon}}{h}\right)^{1/\beta}$. Let $P_1 := \{v_1 = \mu_1, v_2, \dots, v_{n+1} = \mu_2\}$ be a partition of segment $[\mu_1, \mu_2]$ such that

$$\|v_i - v_{i+1}\| = \frac{\|\mu_1 - \mu_2\|}{n}.$$

Then, we have

$$\|v_i - v_{i+1}\|^\beta = \left(\frac{\|\mu_1 - \mu_2\|}{n}\right)^\beta \leq \left(\frac{\rho_1}{n}\right)^\beta \leq \frac{\bar{\varepsilon}}{h} \leq \frac{\varepsilon_2}{h}.$$

Hence, using the result of Case 1, one gets

$$\mathcal{H}(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_2))$$

$$\begin{aligned}
&\leq \sum_{i=1}^n \mathcal{H} \left(S_{\xi}(\varepsilon_2, \lambda_1, v_i), S_{\xi}(\varepsilon_2, \lambda_1, v_{i+1}) \right) \\
&\leq \rho_1 h(\bar{\varepsilon})^{-1} \sum_{i=1}^n \|v_i - v_{i+1}\|^{\beta} \\
&\leq n \rho_1 h(\bar{\varepsilon})^{-1} \|\mu_1 - \mu_2\|^{\beta} \\
&:= k_2 \|\mu_1 - \mu_2\|^{\beta}.
\end{aligned}$$

Step 3. Let $\varepsilon_2 \in [\bar{\varepsilon}, +\infty)$, $\mu_2 \in V$ and $\lambda_1, \lambda_2 \in U'$, where $U' \subset U$ is a neighborhood of λ_0 with $\text{diam } U' \leq \frac{1}{2}$. We provide an estimation for $\mathcal{H} \left(S_{\xi}(\varepsilon_2, \lambda_1, \mu_2), S_{\xi}(\varepsilon_2, \lambda_2, \mu_2) \right)$ with $\lambda_1 \neq \lambda_2$. Putting

$$\eta := \varepsilon_2 - \bar{\varepsilon}$$

and

$$\delta := (l_1 m^{\alpha_1} + l_2 m^{\alpha_2}) \|\lambda_1 - \lambda_2\|^{\alpha_3}$$

where $\alpha_3 := \min\{\alpha, \alpha\alpha_1, \alpha\alpha_2\}$, we have also two cases.

Case 1: $\delta \leq \eta$. Then $\varepsilon_2 - \delta \geq \varepsilon_2 - \eta = \bar{\varepsilon}$. For all $x_1 \in S_{\xi}(\varepsilon_2 - \delta, \lambda_1, \mu_2)$, then $x_1 \in K(\lambda_1)$ and $f(x_1, y, \mu_2) + \varepsilon_2 - \delta \geq 0$ for all $y \in K(\lambda_1)$. By (i), there exists $x_2 \in K(\lambda_2)$ such that

$$\|x_1 - x_2\| \leq m \|\lambda_1 - \lambda_2\|^{\alpha}.$$

For each $y_2 \in K(\lambda_2)$, also by (i), there is $y_1 \in K(\lambda_1)$ satisfying $\|y_1 - y_2\| \leq m \|\lambda_1 - \lambda_2\|^{\alpha}$. Since $y_1 \in K(\lambda_1)$, one has

$$\langle \xi, f(x_2, y_2, \mu_2) \rangle + \langle \xi, f(x_1, y_1, \mu_2) \rangle - \langle \xi, f(x_2, y_2, \mu_2) \rangle + \varepsilon_2 - \delta \geq 0.$$

By (iii), we get

$$\begin{aligned}
&|\langle \xi, f(x_1, y_1, \mu_2) \rangle - \langle \xi, f(x_2, y_2, \mu_2) \rangle| \\
&\leq l_1 \|x_1 - x_2\|^{\alpha_1} + l_2 \|y_1 - y_2\|^{\alpha_2} \\
&\leq l_1 m^{\alpha_1} \|\lambda_1 - \lambda_2\|^{\alpha\alpha_1} + l_2 m^{\alpha_2} \|\lambda_1 - \lambda_2\|^{\alpha\alpha_2} \\
&\leq \delta.
\end{aligned}$$

Therefore, $\langle \xi, f(x_2, y_2, \mu_2) \rangle + \varepsilon_2 \geq 0$, that is, $x_2 \in S_{\xi}(\varepsilon_2, \lambda_2, \mu_2)$.

Now, for each $x_0 \in S_{\xi}(\varepsilon_2, \lambda_1, \mu_2)$ and $x_1 \in S_{\xi}(\varepsilon_2 - \delta, \lambda_1, \mu_2)$, we have

$$\begin{aligned}
d(x_0, S_{\xi}(\varepsilon_2, \lambda_2, \mu_2)) &\leq \|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \\
&\leq m \|\lambda_1 - \lambda_2\|^{\alpha} + \|x_1 - x_0\|.
\end{aligned}$$

Because x_1 is arbitrary, one gets

$$d(x_0, S_{\xi}(\varepsilon_2, \lambda_2, \mu_2)) \leq m \|\lambda_1 - \lambda_2\|^{\alpha} + d(x_0, S_{\xi}(\varepsilon_2 - \delta, \lambda_1, \mu_2)),$$

which yields that

$$\begin{aligned}
&\mathcal{H}^* \left(S_{\xi}(\varepsilon_2, \lambda_1, \mu_2), S_{\xi}(\varepsilon_2, \lambda_2, \mu_2) \right) \\
&\leq m \|\lambda_1 - \lambda_2\|^{\alpha} + \mathcal{H}^* \left(S_{\xi}(\varepsilon_2, \lambda_1, \mu_2), S_{\xi}(\varepsilon_2 - \delta, \lambda_1, \mu_2) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq m\|\lambda_1 - \lambda_2\|^\alpha + \mathcal{H}\left(S_\xi(\varepsilon_2, \lambda_1, \mu_2), S_\xi(\varepsilon_2 - \delta, \lambda_1, \mu_2)\right) \\
&\leq m\|\lambda_1 - \lambda_2\|^\alpha + \rho(\bar{\varepsilon})^{-1}|\delta| \\
&\leq m\|\lambda_1 - \lambda_2\|^\alpha + \rho(l_1 m^{\alpha_1} + l_2 m^{\alpha_2})(\bar{\varepsilon})^{-1}\|\lambda_1 - \lambda_2\|^{\alpha_3} \\
&\leq (m\bar{\varepsilon} + \rho(l_1 m^{\alpha_1} + l_2 m^{\alpha_2}))(\bar{\varepsilon})^{-1}\|\lambda_1 - \lambda_2\|^{\alpha_3}.
\end{aligned}$$

Case 2: $\delta > \eta$. Then there is a natural number n satisfying $\frac{\delta}{n} \leq \eta$. Let $\mathbb{P}'$ be a partition of segment $[\lambda_1, \lambda_2]$ with $n + 1$ nodes $u_1, u_2, \dots, u_{n+1}$ satisfying $u_1 = \lambda_1, u_{n+1} = \lambda_2, \|u_i - u_{i+1}\| = \frac{\|\lambda_1 - \lambda_2\|}{n}$. Then

$$\|u_i - u_{i+1}\|^{\alpha_3} \leq \frac{\|\lambda_1 - \lambda_2\|^{\alpha_3}}{n} \leq \frac{\eta}{l_1 m^{\alpha_1} + l_2 m^{\alpha_2}},$$

i.e., $(l_1 m^{\alpha_1} + l_2 m^{\alpha_2})\|u_i - u_{i+1}\|^{\alpha_3} \leq \eta$. Applying Case 1, we get

$$\begin{aligned}
&\mathcal{H}\left(S_\xi(\varepsilon_2, \lambda_1, \mu_2), S_\xi(\varepsilon_2, \lambda_2, \mu_2)\right) \\
&\leq \sum_{i=1}^n \mathcal{H}\left(S_\xi(\varepsilon_2, u_i, \mu_2), S_\xi(\varepsilon_2, u_{i+1}, \mu_2)\right) \\
&\leq n\left(m + \frac{\rho(l_1 m^{\alpha_1} + l_2 m^{\alpha_2})}{\bar{\varepsilon}}\right)\|u_i - u_{i+1}\|^{\alpha_3} \\
&\leq \left(m + \frac{\rho(l_1 m^{\alpha_1} + l_2 m^{\alpha_2})}{\bar{\varepsilon}}\right)n^{1-\alpha_3}\|\lambda_1 - \lambda_2\|^{\alpha_3} \\
&:= k_3\|\lambda_1 - \lambda_2\|^{\alpha_3}.
\end{aligned}$$

Step 4. Combining the results of the preceding three steps, we obtain

$$\begin{aligned}
&\mathcal{H}\left(S_\xi(\varepsilon_1, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_2, \mu_2)\right) \\
&\leq \mathcal{H}\left(S_\xi(\varepsilon_1, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_1)\right) \\
&\quad + \mathcal{H}\left(S_\xi(\varepsilon_2, \lambda_1, \mu_1), S_\xi(\varepsilon_2, \lambda_1, \mu_2)\right) \\
&\quad + \mathcal{H}\left(S_\xi(\varepsilon_2, \lambda_1, \mu_2), S_\xi(\varepsilon_2, \lambda_2, \mu_2)\right) \\
&\leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha_\delta},
\end{aligned}$$

which finishes the proof. $\square$

We now state a result on Hölder continuity of the map S .

Theorem 3.2. Assume that the existence of solutions and (i), (ii), (iii) are as in Theorem 3.1. Assume further that

(iv) For $x \in K(U)$ and $\mu \in M$, the map $y \mapsto f(x, y, \mu)$ is C -convexlike on $K(\Lambda)$.

Then, for any $\bar{\varepsilon} > 0$, there exist open neighborhoods $N'(\lambda_0)$ of λ_0 and $N'(\mu_0)$ of μ_0 such that the mapping S is Hölder continuous on $[\bar{\varepsilon}, +\infty) \times N'(\lambda_0) \times N'(\mu_0)$, i.e.,

$$\mathcal{H}(S(\varepsilon_1, \lambda_1, \mu_1), S(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha_\delta},$$

whenever $(\varepsilon_i, \lambda_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N'(\lambda_0) \times N'(\mu_0)$, $i = 1, 2$ and k_1, k_2, k_3 are positive and depend on $\bar{\varepsilon}, m, \alpha, h, \beta$, etc.

Proof. We see that B_e^* is a weak* compact set and in the Theorem 3.1, the system of $\{N(\bar{\xi})\}_{\bar{\xi} \in B_e^*}$ is an open covering of B_e^* . Therefore, there is a finite number of points $\xi_i \in B_e^*$ ($i = 1, \dots, n$) such that

$$B_e^* \subset \bigcup_{i=1}^n N(\xi_i).$$

Then, for any $\xi \in B_e^*$, there exists $i_0 \in \{1, \dots, n\}$ such that $\xi \in N(\xi_{i_0})$. Let us put $N'(\mu_0) = \bigcap_{i=1}^n N_{\xi_i}(\mu_0)$ and $N'(\lambda_0) = \bigcap_{i=1}^n N_{\xi_i}(\lambda_0)$. Take arbitrarily $(\lambda, \mu) \in N'(\lambda_0) \times N'(\mu_0)$, from the construction of the neighborhoods $N'(\lambda_0)$ and $N'(\mu_0)$, we have $(\lambda, \mu) \in N_{\xi_{i_0}}(\lambda_0) \times N_{\xi_{i_0}}(\mu_0)$. So, $N'(\lambda_0)$ and $N'(\mu_0)$ are desired neighborhoods of λ_0 and μ_0 , respectively. By (iv) and Lemma 2.7, one has

$$S(\varepsilon, \lambda, \mu) = \bigcup_{\bar{\xi} \in B_e^*} S_{\bar{\xi}}(\varepsilon, \lambda, \mu).$$

For all $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N'(\lambda_0) \times N'(\mu_0)$, it is claimed that

$$\mathcal{H}(S(\varepsilon_1, \lambda_1, \mu_1), S(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

Indeed, for each $x_1 \in S(\varepsilon_1, \lambda_1, \mu_1) = \bigcup_{\bar{\xi} \in B_e^*} S_{\bar{\xi}}(\varepsilon_1, \lambda_1, \mu_1)$, there is $\hat{\xi} \in B_e^*$ such that $x_1 \in S_{\hat{\xi}}(\varepsilon_1, \lambda_1, \mu_1)$. As $S_{\hat{\xi}}(\varepsilon_2, \lambda_2, \mu_2) \subset S(\varepsilon_2, \lambda_2, \mu_2)$ and applying Theorem 3.1, one has

$$\begin{aligned} d(x_1, S(\varepsilon_2, \lambda_2, \mu_2)) &\leq d(x_1, S_{\hat{\xi}}(\varepsilon_2, \lambda_2, \mu_2)) \\ &\leq \mathcal{H}^*(S_{\hat{\xi}}(\varepsilon_1, \lambda_1, \mu_1), S_{\hat{\xi}}(\varepsilon_2, \lambda_2, \mu_2)) \\ &\leq \mathcal{H}(S_{\hat{\xi}}(\varepsilon_1, \lambda_1, \mu_1), S_{\hat{\xi}}(\varepsilon_2, \lambda_2, \mu_2)) \\ &\leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha\delta}. \end{aligned}$$

where k_1, k_2, k_3 are positive and depend on $\bar{\varepsilon}, l, \alpha, h, \beta$, etc. Therefore,

$$\mathcal{H}^*(S(\varepsilon_1, \lambda_1, \mu_1), S(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

By the same way, we also obtain

$$\mathcal{H}^*(S(\varepsilon_2, \lambda_2, \mu_2), S(\varepsilon_1, \lambda_1, \mu_1)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

As a result, we have

$$\mathcal{H}(S(\varepsilon_1, \lambda_1, \mu_1), S(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^\beta + k_3\|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

Hence, the proof is complete. $\square$

Remark 3.3. It is worth noting that in Theorem 3.2, we have relaxed the concavity in Theorem 3.1 of¹³ and in Theorem 3.2 of⁹ to the C -lower level quasiconcavity.

The following example shows that Theorem 3.2 can be applicable but the results presented in^{9,13} cannot.

Example 3.4. Let $A = X = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}_+^2, e = (1, 1) \in \text{int}C, \Lambda = M = [0, 1], K(\lambda) = [-1, 1]$ and $f(x, y, \mu) = (x^2 + \mu, x^2 + \mu)$. Then all assumptions of Theorem 3.2 are satisfied. The approximate solution set is $S(\mu) = [-1, 1]$, which is Hölder continuous. However, the results in^{9,13} do not work since the function f is not concave in the first component.

The following corollary is directly derived from Theorem 3.2.

Corollary 3.5. Assume that

- (i) K is m -Lipschitz continuous on Λ ;
- (ii) for each $y \in K(\Lambda)$ and $\mu \in M, (x, \varepsilon) \mapsto f(x, y, \mu) + \varepsilon e$ is C -lower level quasiconcave on $K(\Lambda) \times (0, +\infty)$;
- (iii) f is (l_1, l_2, h) -Lipschitz continuous on $K(\Lambda) \times K(\Lambda) \times M$;
- (iv) for $\mu \in M$ and $x \in K(\Lambda), y \mapsto f(x, y, \mu)$ is C -convex-like on $K(\Lambda)$.

Then, for any $\bar{\varepsilon} > 0$, there exist open neighborhoods $N'(\lambda_0)$ of λ_0 and $N'(\mu_0)$ of μ_0 such that the mapping S is Lipschitz continuous on $[\bar{\varepsilon}, +\infty) \times N'(\lambda_0) \times N'(\mu_0)$, i.e.,

$$\mathcal{H}(S(\varepsilon_1, \lambda_1, \mu_1), S(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\| + k_3\|\lambda_1 - \lambda_2\|,$$

whenever $(\varepsilon_i, \lambda_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N'(\lambda_0) \times N'(\mu_0), i = 1, 2$ and k_1, k_2, k_3 are positive and depend on $\bar{\varepsilon}, m, h, l_1, l_2$, etc.

In the case of $K(\lambda) = K$ for all λ , by using the same technique as that of Theorem 3.2, we also obtain the following result.

Corollary 3.6. For (WEP), where $K(\lambda) \equiv K$. Suppose that $S(\varepsilon, \mu)$ is nonempty in a neighborhood of the considered point (ε_0, μ_0) . Assume further that

- (i) For each $y \in K$ and $\mu \in M, (x, \varepsilon) \mapsto f(x, y, \mu) + \varepsilon e$ is C -lower level quasiconcave on $K \times (0, +\infty)$;
- (ii) for each $x, y \in K, \mu \mapsto f(x, y, \mu)$ is h -Lipschitz continuous on M ;
- (iii) for $\mu \in M$ and $x \in K, y \mapsto f(x, y, \mu)$ is C -convex-like on K .

Then, for any $\bar{\varepsilon} > 0$, there exist open neighborhood $N'(\mu_0)$ of μ_0 such that the approximate solution mapping S is Lipschitz continuous on $[\bar{\varepsilon}, +\infty) \times N'(\mu_0)$, i.e.,

$$\mathcal{H}(S(\varepsilon_1, \mu_1), S(\varepsilon_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|,$$

whenever $(\varepsilon_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N'(\mu_0), i = 1, 2$ and k_1, k_2 are positive, depend on $\bar{\varepsilon}, h$.

If $Y = \mathbb{R}, C = \mathbb{R}_+, e = 1$, then Corollary 3.6 reduces a special result as follows.

Corollary 3.7. For (WEP), where $K(\lambda) \equiv K, Y = \mathbb{R}, C = \mathbb{R}_+, e = 1$. Suppose that $S(\varepsilon, \mu)$ is nonempty in a neighborhood of the considered point (ε_0, μ_0) . Furthermore, assume that the following conditions hold.

- (i) For each $y \in K$ and $\mu \in M, (x, \varepsilon) \mapsto f(x, y, \mu) + \varepsilon e$ is $\mathbb{R}_+$ -lower level quasiconcave on $K \times (0, +\infty)$;
- (ii) for each $x, y \in K, \mu \mapsto f(x, y, \cdot)$ is h -Lipschitz continuous on M .

Then, for any $\bar{\varepsilon} > 0$, there exist open neighborhood $N'(\mu_0)$ of μ_0 such that the approximate solution mapping S is Lipschitz continuous on $[\bar{\varepsilon}, +\infty) \times N'(\mu_0)$, i.e.,

$$\mathcal{H}(S(\varepsilon_1, \mu_1), S(\varepsilon_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|,$$

whenever $(\varepsilon_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N'(\mu_0), i = 1, 2$ and k_1, k_2 are positive and depend on $\bar{\varepsilon}, h$.

Remark 3.8. For this special case, (WEP) in Corollary 3.7 reduces to (PSEP) considered in²⁵. In this case, Corollary 3.7 is an improvement of [²⁵, Theorem 2.3]. To be more precise, the upper semicontinuity of f in the first component is not necessary and the concavity of f in the third component has been replaced by the $\mathbb{R}_+$ -lower level quasiconcavity of f .

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

AUTHOR CONTRIBUTIONS

Le Xuan Dai: Methodology, validation, supervision, writing-review & editing.

Tran Quoc Duy: Investigation, visualization, writing-editing support.

Tran Ngoc Tam: Investigation, literature review, theoretical framework, examples and counterexamples.

All authors have read and approved the final manuscript.

LIST OF ABBREVIATIONS

WEP: Vector Equilibrium Problem.

PSEP: Parametric Scalar Equilibrium Problem.

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ỔN ĐỊNH HÖLDER CHO CÁC BÀI TOÁN CÂN BẰNG VECTƠ THAM SỐ

Lê Xuân Đại^{1,2}, Trần Quốc Duy³, Trần Ngọc Tâm^{4,*}



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¹Khoa Khoa học Ứng dụng, Trường Đại học Bách khoa TP. Hồ Chí Minh (HCMUT), 268 Lý Thường Kiệt, Phường Diên Hồng, TP. Hồ Chí Minh

²Đại học Quốc gia Thành phố Hồ Chí Minh (VNU-HCM), Việt Nam

³Bộ môn Toán học, Trường Đại học FPT, Cần Thơ, Việt Nam

⁴Khoa Toán học, Trường Khoa học Tự nhiên, Đại học Cần Thơ, Cần Thơ, Việt Nam

Liên hệ

Trần Ngọc Tâm, Khoa Toán học, Trường Khoa học Tự nhiên, Đại học Cần Thơ, Cần Thơ, Việt Nam

Email: tntam@ctu.edu.vn

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TÓM TẮT

Ổn định là một chủ đề cơ bản và quan trọng trong lý thuyết tối ưu. Mục tiêu chính của lĩnh vực này là nghiên cứu sự thay đổi của nghiệm đối với các bài toán khi dữ liệu của bài toán có những thay đổi nhỏ. Do đó, tính ổn định đối với các bài toán liên quan đến tối ưu hóa mang ý nghĩa cả về lý thuyết lẫn thực tiễn, và được xem như một công cụ hữu ích trong phân tích hậu tối ưu. Thông thường, ổn định có thể được hiểu theo hai cách: (1) cách thứ nhất liên quan đến tính nửa liên tục trên, nửa liên tục dưới và tính liên tục của ánh xạ nghiệm đối với các bài toán tham số, được gọi là ổn định định tính; (2) cách thứ hai bao gồm tính liên tục Hölder/Lipschitz của ánh xạ nghiệm của các bài toán, được gọi là ổn định định lượng. Đối với các bài toán tham số mà tham số bị nhiễu trong không gian tham số, nhiều nhà nghiên cứu đã thiết lập các điều kiện ổn định khác nhau, dựa trên tính liên tục của các ánh xạ mục tiêu và ánh xạ ràng buộc, cùng với các tính chất lồi và đơn điệu của chúng. Trong bài báo này, chúng tôi nghiên cứu các bài toán cân bằng vectơ tham số trong không gian định chuẩn và hướng tới việc cải thiện một số kết quả gần đây. Để đạt được điều này, trước tiên chúng tôi giới thiệu một khái niệm mở rộng của tính tựa lồi theo nón, cụ thể là tính tựa lồi mức dưới theo nón. Đồng thời, chúng tôi khảo sát các điều kiện cần và đủ tương ứng. Một ví dụ được đưa ra để minh họa rằng điều kiện mới này yếu hơn điều kiện tính tựa lồi theo nón. Sau đó, chúng tôi sử dụng điều kiện mới này cùng với các tính chất Hölder của ánh xạ ràng buộc và ánh xạ mục tiêu để thiết lập các điều kiện đủ cho tính ổn định theo nghĩa liên tục Hölder của ánh xạ nghiệm đối với các bài toán đang xét, thông qua phương pháp tuyến tính hóa vô hướng. Một trường hợp đặc biệt của bài toán đang xét cũng được thảo luận. Các kết quả trong bài báo này cải thiện và mở rộng các kết quả hiện có trong tài liệu trước đây.

Từ khoá: Vector equilibrium problem, Hölder continuity, parametric problem, approximate solution

Trích dẫn bài báo này: Đại L X, Duy T Q, Tâm T N. ỔN ĐỊNH HÖLDER CHO CÁC BÀI TOÁN CÂN BẰNG VECTƠ THAM SỐ. *VNUHCM J. Eng. Technol.* 2026; 9(3): 3326-3338.