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Research Article

Stability for vector equilibrium problems

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ABSTRACT

This paper focuses on the stability analysis of Minty and Stampacchia vector equilibrium problems, particularly in scenarios where both the feasible set and the objective map are subject to perturbations. The research utilizes advanced mathematical tools such as cone-semicontinuity and generalized level closedness properties of objective maps to derive significant upper stability results for these vector equilibrium problems. Additionally, the study explores lower stability outcomes by applying generalized cone-convexity assumptions on objective maps, notably without the need for traditional monotonicity properties, which are often restrictive in practical applications.

The findings in this paper represent a substantial contribution to the field of vector equilibrium problems by broadening the scope of existing stability theories. This work overcomes some of the limitations of prior research, which typically required strict continuity of objective maps or the solvability of auxiliary problems. By relaxing these conditions, the paper offers a more robust framework for analyzing the stability of vector equilibrium problems, making the results applicable to a wider array of practical situations, including those in physics, engineering, economics, and social network analysis.

Ultimately, this paper advances the understanding of stability in vector equilibrium problems, providing a foundation for future research and potential applications in various scientific and engineering disciplines. The theoretical developments presented here not only enhance the mathematical modeling of complex systems but also contribute to the practical implementation of these models in real-world scenarios. Furthermore, several illustrative examples are provided to demonstrate the applicability and effectiveness of the obtained stability results. These examples highlight the flexibility of the proposed approach and emphasize its potential usefulness in addressing complex equilibrium models arising in optimization, decision science, and multi-criteria analysis.

Key words: Vector equilibrium problem, Upper stability, Lower stability, Cone-continuity, Cone-convexity

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INTRODUCTION

The scalar equilibrium problem model plays a central role in nonlinear analysis because its setting encompasses several important problems in physics, engineering, economics such as traffic equilibrium, transportation network congestion, optimal shape design, electric power markets, social and economic networks, and so forth. Motivated by the pioneer work of Giannessi[1], which extended classical variational inequalities to the case of vector-valued map, many mathematicians have extended the scalar equilibrium problem to the case of vector-valued objective map, known as vector equilibrium problem. We refer the reader to[2–7] and references therein for more information.

The theory on existence conditions and solution methods to equilibrium problems and their generalizations has attracted immense attention from researchers[3,4,8–12]. A relatively new but rapidly developing topic is the stability property of solutions including upper/lower semi-continuity properties in the sense of Berge and Hausdorff. Inspired by the ideas of Cheng and Zhu[13] on the stability for weak vector variational inequality, Gong[14] studied the continuity of the solution maps to the mixed parametric monotone weak vector equilibrium problems in Hausdorff topological vector spaces. Utilizing a density result and scalarization technique, Gong and Yao[15] established the lower semi-continuity of efficient solutions maps to parametric generalized systems with monotone bifunctions in real locally convex Hausdorff topological vector spaces. With the aim of extending the results in[14,15], Li and Fang[16] introduced a relaxed assumption of strict cone-maps concerning monotonicity properties and utilized it to investigate the lower semi-continuity of weak vector solutions maps to a parametric generalized Ky Fan inequality. A drawback of using the monotonicity property on objective is that the set of solutions may be a singleton, and hence the applicability of obtained results in above works is somewhat restricted. Taking this observation into account, Zhang et al.[17] proposed Hölder-related

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assumptions to study the lower semi-continuity for the weak efficient solution maps to parametric vector equilibrium problems without employing any assumptions regarding monotonicity. Admittedly, this assumption is not natural and also hard to apply to practical situations because it requires to know the information concerning solution sets of the reference problems. Furthermore, a common point of these works is that the objective maps were always imposed to be continuous. Semi-continuity properties of solution maps to bilevel vector equilibrium problems have been recently investigated in [18,19], herein to establish the lower semi-continuity of solution maps to the first level problems it was implicitly assume that solution sets to auxiliary problems were nonempty, which could be a very restrictive assumption in many special cases such as optimization problems, variational inequalities.

This paper is devoted to the investigation upper and lower continuity of solution maps to perturbed vector equilibrium problems, obtained by perturbing the constraint sets and the objective maps. In light of cone-continuity concepts, proposed in [20], along with generalized level closedness properties of the objective we present upper stability conditions for the reference problems. For the lower stability of such problems, we only require additional assumptions related to the generalized convexity, and do not impose any monotonicity properties and the solvability of auxiliary problems.

The paper layout is organized as follows. Section 2 contains preliminary terminology and notation used in the paper. In Section 3, conditions on objective maps and set-valued ones defining feasible region are given to establish the upper stability for Minty and Stampacchia vector equilibrium problems. Section 4 is devoted to the lower continuity and continuity of weak efficient solutions maps to the reference problems.

PRELIMINARIES

In this section, we first give the preliminary terminology and notations, which will be used throughout the paper. Let $\mathscr{X}$, Y and T be Hausdorff topological vector spaces, and X be a nonempty subset of $\mathscr{X}$. The set of real numbers (respectively, nonnegative real numbers and natural numbers) is denoted by R (respectively, R_+ and N). For a nonempty subset A and B of Y, we denote the topological interior of A by int A; the product of $t \in R$ and A by $tA := \{ta \mid a \in A\}$; the Minkowski sum and difference of A and B by

$$A + B := \{a + b \mid a \in A, b \in B\},$$
$$A - B := \{a - b \mid a \in A, b \in B\}.$$

A nonempty set $C \subset Y$ is called a cone whenever $tC \in C$ for all $t \in R_+$. The cone C is said to be pointed if $C \cap (-C) = \{0_y\}$; convex if $C + C = C$; solid if $intC \neq \varnothing$. In this paper, we always assume that C is a pointed convex closed and solid cone in Y. For $y_1, y_2 \in Y$, we define binary relations $\leq, <,$, and $\nless$ on Y as follows:

$$y_1 \leq y_2 :\Leftrightarrow y_2 - y_1 \in C,$$

$$y_1 < y_2 :\Leftrightarrow y_2 - y_1 \in intC,$$

$$y_1 \nleq y_2 :\Leftrightarrow y_2 - y_1 \notin C,$$

$$y_1 \nless y_2 : \Leftrightarrow y_2 - y_1 \notin \text{intC}.$$

For $a \in Y$, a level set of a vector-valued map $g : \mathscr{X} \to Y$ is defined as

$$lev_{\nless a} g := \{x \in \mathscr{X} \mid g(x) \nless a\}.$$

We now introduce the concepts of upper and lower semi-continuity for vector-valued maps, which are generalizations of classical upper and lower semi-continuity of real-valued functions.

Definition 2.1. [20] For a given $\overline{x} \in X$, a vector-valued map $g : X \to Y$ is said to be (a) C-lower semicontinuous (C-lsc) at $\overline{x}$ if, for any neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ such that

$$g(x) \in g(\overline{x}) + V + C, \forall x \in U.$$

(b) C-upper semicontinuous (C-usc) at $\overline{x}$ if, $-g$ is C-usc at $\overline{x}$.

(c) C-continuous at $\overline{x}$ if it is both C-usc and C-lsc at $\overline{x}$.

In what follows, we say that a map holds a certain property on a subset $M \subset \mathscr{X}$ if so does it at each point of M. When $M = \mathscr{X}$, we omit "in $\mathscr{X}$ " in the statement.

We now recall some characterizations for the cone-lower semi-continuity of a vector-valued map.

Proposition 2.2. [20] Let $g : X \to Y$ be a vector-valued map. The following assertions are equivalent.

(i) For each $a \in Y$, $lev_{\nless a}\, g$ closed.

(ii) g is C-lower semicontinuous.

(iii) For each $\overline{x} \in X$ and $c \in int\, C$, there is an neighborhood U of $\overline{x}$ such that $g(x) \in g(\overline{x}) - c + int\, C$, for all $x \in U$.

We have also an analogous result for the C-upper semi-continuity property of a vectorvalued map.

Proposition 2.3. Let $g : X \to Y$ be a vector-valued map. Then, the following assertions are equivalent.

(i) For each $a \in Y$, $lev_{\nless a}\, g$ closed.

(ii) g is C-lower semicontinuous.

(iii) For each $\overline{x} \in X$ and $c \in int\ C$, there is a neighborhood U of $\overline{x}$ such that $g(x) \in g(\overline{x}) + c - int\ C$, for all $x \in U$.

Definition 2.4. [2, Definition 2.28] A vector-valued map $g : X \to Y$ is said to be lower hemicontinuous (respectively, C-upper hemicontinuous) if for all $x, y \in X$, the vector-valued function $t \mapsto g(x + t(y - x))$ with $t \in [0,1]$, is C -lsc (respectively, C-usc) at t = 0.

Definition 2.5. [2, Definition 2.13] Let M convex subset of $\mathscr{X}$. A vector-valued *map* $g : \mathscr{X} \to Y$ is said to be (a) C -quasiconcave on M if for each $y \in Y$, and for all $x_1, x_2 \in M, t \in [0,1]$, $g(x_1) \in y + C, g(x_2) \in y + C$ will imply $g(x_t) \in y + C$, where $x_t = tx_1 + (1-t)x_2$;(b) strictly C-quasiconcave on M if for each $y \in Y$, and for all $x_1, x_2 \in M, x_1 \neq x_2$, $t \in (0,1)$, $g(x_1) \in y + C, g(x_2) \in y + C$ will imply $g(x_t) \in y + intC$,where $x_t = tx_1 + (1-t)x_2$.

In a particular case in which $Y = R, C = R^+$, we get the definitions of quasiconcave and strictly quasiconcave functions in the usual sense.

Definition 2.6. [21, Definitions 3.1.1 and 3.1.7] A set-valued map F acting from X into Y is said to be

(a) upper continuous (u.c.) at $x_0 \in X$ if for any open superset U of $F(x_0)$, there is a neighborhood N of x_0 such that $F(x) \subset U$ for all $x \in N$.

(b) lower continuous (l.c.) at $x_0 \in X$ if for any open subset U of Y with $F(x_0) \cap U \neq \varnothing$, there is a neighborhood N of x_0 such that for all $x \in N, F(x) \cap U \neq \varnothing$.

(c) continuous at $x_0 \in X$ if it is both u.c. and l.c. at x_0.

(d) closed at x_0 if for every net $\{(x_i, y_i)\}_{i \in I} \subset gph\ F$ converging to (x_0, y_0) one has that $y_0 \in F(x_0)$.

(e) compact at $\overline{x}$ if for every net $\{(x_i, y_i)\}_{i \in I} \subset gph\ F$ with $x_i \to x_0, (y_i)_{i \in I}$ admits a subnet converging to some $y_0 \in F(x_0)$.We recall the concepts of Painlevé-Kuratowski lower and upper limits for a sequence of nonempty subsets (X_n) of X as follows

$$LiX_n := \{x \in \mathscr{X} \mid x = \lim_{n\to\infty} x_n, x_n \in X_n\},$$
$$LsX_n := \left\{x \in \mathscr{X} \mid x = \lim_{k\to\infty} x_k, x_k \in X_{n,k,\{n_k\}}\ subsequence\ of\ \{n\}\right\}$$

Lemma 2.7. (See e.g. [21, Proposition 3.1.6 and Proposition 3.1.9]) Let $F : \mathscr{X} \rightrightarrows Y$ be a set-valued map and $\overline{x} \in \mathscr{X}$. Then, the following assertions hold.

(a) If $F(\overline{x})$ is compact, then F is u.c. at $\overline{x}$ if and only if for any sequence $\{x_n\} \subset \mathscr{X}$ with $x_n \to \overline{x}$ and $y_n \in F(x_n)$, there is a subsequence $\{y_{n_k}\}$ that converges to some $\overline{y} \in F(\overline{x})$.

(b) F is l.c. at $\overline{x}$ if and only if, for any sequence $(x_n) \subset X$ with $x_n \to \overline{x}$ and $\overline{y} \in F(\overline{x})$, there exists a sequence $\{y_n\}$ of $F(x_n)$ such that $y_n \to \overline{y}$.

(c) F is l.c. at $\overline{x}$ if and only if, for any sequence $\{x_n\} \subset X$ with $x_n \to \overline{x}$, one has $F(\overline{x}) \subset\ Li\ F(x_n)$

Lemma 2.8. [22] Let $\{X_n\}$ be a sequence of subsets of X. Suppose that

(i) X_n is convex with $intX_n \neq \varnothing$ for all n;

(ii) there exists a nonempty compact set $X \subset X$ such that $X \subset LiX_n$.

Then,

$$X \subset \bigcup_{m\geq 1} \bigcap_{n\geq m} int\ X_n.$$

UPPER STABILITY FOR MINTY AND STAMPACCHIA VECTOR EQUILIBRIUM PROBLEMS

Let B be a convex cone in Y, and $f : X \times X \to Y$ be a vector-valued map. The Minty vector equilibrium problem is defined as follows: (MEP) find $\overline{x} \in X$ such that $f(y, \overline{x}) \notin B, \forall y \in X$.

We are interested in the parametric Minty vector equilibrium problem under perturbations in terms of perturbing the feasible set X and the objective map f by a parameter λ varying on a subset Λ of T.

$(PMEP)_\lambda$ find $\overline{x} \in X$ such that $f(y, \overline{x}, \lambda) \notin B, \forall y \in K(\lambda)$,

where $K : \Lambda \rightrightarrows X$ is a set-valued map, and $f : X \times X \times \Lambda \to Y$ is a vector-valued map. Instead of writing $((PMEP)_\lambda \mid \lambda \in \Lambda\}$ for the family of Minty vector equilibrium problems, we will simply write (PMEP) in the sequel.

A vector $x \in K(\lambda)$ is said to be a weak efficient solution to (PMEP) if

$$f(y, x, \lambda) \notin intC,\ \forall y \in K(\lambda).$$

Let $\Omega : \Lambda \rightrightarrows X$ be a set-valued map such that $\Omega(\lambda)$ is the set of weak efficient solutions to the problem (PMEP) for $\lambda \in \Lambda$. In this paper, we focus on the continuity property of the solution map Ω and we always assume that $\Omega(\lambda)$ is nonempty for each λ in a neighborhood of the reference point. The existence results for such problems can be found in [2,4,10] and the references therein.

We now study the upper continuity and closedness of the weak efficient solution maps to (PMEP).

Theorem 3.1. For the problem (PMEP), assume that

(i) K is closed and l.c. on Λ;

(ii) f is C-lsc on $X \times X \times \Lambda$.

Then, the solution map Ω is closed on Λ. Furthermore, if K is compact on Λ, then Ω is u.c. on Λ.

Proof. Let $\overline{\lambda}$ be arbitrary in Λ. We first claim that Ω is closed at $\overline{\lambda}$. Let $\{(\lambda_n, x_n)\}$ be a sequence in gph Ω converging to $\left(\overline{\lambda}, \overline{x}\right)$ with $\overline{x} \in X$. Since K is closed at $\overline{\lambda}$, the point $\overline{x}$ belong to $K\left(\overline{\lambda}\right)$. We show that $\overline{x} \in$

$\Omega\left(\overline{\lambda}\right)$. Indeed, if otherwise, there exists $\overline{y} \in K\left(\overline{\lambda}\right)$ such that

$$f(\overline{y}, \overline{x}, \overline{\lambda}) \in int\, C.$$

It follows from the lower continuity of K at $\overline{\lambda}$ that there is a sequence $\{y_n\}$ with $y_n \in K(\lambda_n)$ and $y_n \to \overline{y}$. Since $x_n \in \Omega(\lambda_n)$,

$$f(y_n, x_n, \lambda_n) \notin intC.$$

For every neighborhood B of the origin in Y, we can find a balanced neighborhood B_1 of θ_Y, i.e., $-B_1 = B_1$, such that $B_1 \subset B$. The C-lower semicontinuity of f at $(\overline{y}, \overline{x}, \overline{\lambda})$ ensures the existence of $k \in N$ such that, for each $n \geq k$, we have

$$f(y_n, x_n, \lambda_n) \in f(\overline{y}, \overline{x}, \overline{\lambda}) + B_1 + C.$$

This together with the balance property of B_1, we get

$$f(\overline{y}, \overline{x}, \overline{\lambda}) \in f(y_n, x_n, \lambda_n) - B_1 - C$$
$$= f(y_n, x_n, \lambda_n) + B_1 - C.$$

From above and taking into account the convexity of C, we obtain

$$f(\overline{y}, \overline{x}, \overline{\lambda}) \in Y \setminus int\, C + B_1 + C$$
$$\subset Y \setminus int\, C + B_1$$
$$\subset Y \setminus int\, C + B,$$

which yields that $f(\overline{y}, \overline{x}, \overline{\lambda}) \in Y \setminus int\, C + B$. Due to the arbitrary choice of B and closedness of $Y \setminus intC$, we conclude that $f(\overline{y}, \overline{x}, \overline{\lambda}) \in Y \setminus int\, C + B$, which is a contradiction. Therefore, $\overline{x} \in \Omega(\overline{\lambda})$, i.e., Ω is closed at $\overline{\lambda}$.

For the second assertion, we suppose to the contrary that for some $\overline{\lambda} \in \Lambda$, there are an open set U of $\Omega\left(\overline{\lambda}\right)$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \overline{\lambda}$ such that $x_n \in \Omega(\lambda_n) \setminus U$ for all $n \in N$. By the compactness of K at $\overline{\lambda}$, one can assume without loss of generality that (x_n) approaches some point $\overline{x}$ in $K\left(\overline{\lambda}\right)$. Using the same arguments as in the first part of the proof, we also obtain that $\overline{x} \in \Omega(\overline{\lambda})$, which furnishes a contradiction as $x_n \notin U$, for all n. Therefore, the solution map Ω is u.c. on Λ. The proof is complete.

The C-lower semicontinuity of Theorem 3.1 can be relaxed to the closedness of the level set $lev_{\not\geq 0Y} f$ demonstrated as in the following result.

Corollary 3.2. For the problem (PMEP), assume that

(i) K is closed and l.c. on Λ;

(ii) $lev_{\not\geq 0Y} f$ is closed on $X \times X \times \Lambda$.

Then, the solution map Ω is closed on Λ. Furthermore, if K is compact on Λ, then Ω is u.c. on Λ.

The proof of Corollary 3.2 is similar to that of Theorem 3.1.

Taking into account techniques in the proof of Theorem 3.1, we have the following results concerning the upper continuity property for a solution map to a Stampacchia vector equilibrium problem, which is stated as follows:

(PSEP) find $\overline{x} \in K(\lambda)$ such that $f(\overline{x}, y, \lambda) \notin -int\, C$, $\forall y \in K(\lambda)$.

For each $\lambda \in \Lambda$, we denote $\Psi(\lambda)$ the set of weak efficient solutions to the problem (PSEP) corresponding to λ.

Theorem 3.3. For the problem (PSEP), assume that

(i) K is closed and l.c. on Λ;

(ii) f is C-usc on $X \times X \times \Lambda$.

Then, the solution map Ψ is closed on Λ. Furthermore, if K is compact on Λ, then Ψ is u.c. on Λ.

Corollary 3.4. For the problem (PSEP), assume that

(i) K is closed and l.c. on Λ;

(ii) $lev_{\not\leq 0Y} f$ is closed on $X \times X \times \Lambda$.

Then, the solution map Ψ is closed on Λ. Furthermore, if K is compact on Λ, then Ψ is u.c. on Λ.

Remark 3.5. Let $\varphi : \Lambda \times K(\Lambda) \times K(\Lambda) \to Y$ and $\psi : \Lambda \times K(\Lambda) \to Y$ be vector-valued maps. For each $\lambda \in \Lambda$, and for all $x, y \in K(\lambda)$ let

$$f(x, y, \lambda) = \varphi(\lambda, x, y) + \psi(\lambda, y) - \psi(\lambda, x)$$

then (PSEP) collapses to the parametric weak vector equilibrium problem considered by Gong [14]. Corollary 3.4 is an improvement of Theorem 3.1 in [14] due to the following reasons. First, it does not require the monotonicity of $\varphi(\lambda, \cdot, \cdot)$ and convexity of $\psi(\lambda, \cdot) + \varphi(\lambda, x, \cdot)$. Second, the continuity of objective map is now relaxed to the closedness of the set $lev_{\not\leq 0Y} f$. Finally, the following example is an illustrate the applicable of Corollary 3.4 while Theorem 3.1 in [14] does not work.

Example 3.6. Let $X = T = R, \Lambda = [0,1], Y = \mathbb{R}^2, C = \mathbb{R}^2_+, K(\lambda) \equiv [0,1]$, and

$$f(x,y,\lambda) = \begin{cases} \left((cos(x-y)+1)sin^2\left(\frac{1}{\lambda}\right), (x-y+1)sin^2\left(\frac{1}{\lambda}\right)\right), & if\ \lambda \neq 0 \\ ((cos(x-y)+1), x-y+1), & if\ \lambda = 0 \end{cases}$$

It is obvious that K is compact, and f is R^2_+-upper semicontinuous. Hence, all assumptions in Corollary 3.4 are satisfied. By direct computations, we get $\Psi(\lambda) = [0,1]$ for all $\lambda \in \Lambda$, which is u.c. and compact-valued. However, since f is not continuous, Theorem 3.1 in [14] is not applicable in this case.

Remark 3.7. It is worth noting that, in Theorems 3.1 and 3.3 we have to impose the condition related to the continuity property of objective mapping at the pair (x, y), which is very restrictive in practical situations, for instance, in the case of $f(x,y) = \langle T(x), x-y\rangle$ where $T \in L(\mathscr{X}, Y)$, the set of all linear continuous mapping from $\mathscr{X}$ into Y. Therefore, it is desirable to study alternate assumptions to overcome this drawback.

Theorem 3.8. For the problem (PMEP), assume that
(i) K is a convex-compact-valued map satisfying int $K(\lambda) \neq \varnothing$ for all $\lambda \in \Lambda$;
(ii) K is closed and l.c. on Λ;
(iii) For each $y \in X, f(y,\cdot,\cdot)$ is C-lsc on $X \times \Lambda$.
Then, the solution map Ω is closed on Λ. In addition, if K is compact on Λ, and for each $(x,\lambda) \in X \times \Lambda, f(\cdot,x,\lambda)$ is C-lower hemicontinuous on X, then Ω is u.c. on Λ.

Proof. Let $\overline{\lambda}$ be arbitrary in Λ. For the first assertion, let a sequence $\{(\lambda_n, x_n)\} \subset gph\,\Omega$ converging to $(\overline{\lambda}, \overline{x})$ with $\overline{x} \in X$. Due to the closedness of K at $\overline{\lambda}$, one has $\overline{x} \in K\left(\overline{\lambda}\right)$. To show that $\overline{x} \in \Omega\left(\overline{\lambda}\right)$, we first verify that $f(y,\overline{x},\overline{\lambda}) \in Y \setminus int\ C$ for all $y \in int\ K\left(\overline{\lambda}\right)$.

Let $\overline{y}$ be an arbitrary point in int $K\left(\overline{\lambda}\right)$. According to Lemma 2.7, the lower continuity of K at $\overline{\lambda}$ implies that $K\left(\overline{\lambda}\right) \subset Li\ K(\lambda_n)$. Because $K\left(\overline{\lambda}\right)$ is compact and $K(\lambda_n)$ is convex with int $K(\lambda_n) \neq \varnothing$, by applying Lemma 2.8 we come to

$$int\ K(\overline{\lambda}) \subset \bigcup_{m\geq 1} \bigcap_{n \geq m} K(\lambda_n).$$

As a result, $\overline{y} \in int\ K(\lambda_n)$ for n sufficiently large, which together with $x_n \in \Omega(\lambda_n)$ gives us

$$f(\overline{y}, x_n, \lambda_n) \notin int\ C.$$

Taking into account the C-lower semicontinuity of $f(\overline{y},\cdot,\cdot)$ at $(\overline{x},\overline{\lambda})$ and utilizing the same arguments as in the first proof of Theorem 3.1, we conclude that $f(\overline{y},\overline{x},\overline{\lambda}) \notin int\ C$ for any $\overline{y} \in int\ K\left(\overline{\lambda}\right)$.

Now, let a point $y \in K\left(\overline{\lambda}\right) \setminus int\ K\left(\overline{\lambda}\right)$ be arbitrary. The convexity of $K\left(\overline{\lambda}\right)$ ensures the existence of a sequence $\{y_n\} \subset int\ K\left(\overline{\lambda}\right)$ converging (along the segment) to y for all n. Consequently, $f\left(y_n,\overline{x},\overline{\lambda}\right) \notin int\ C$ inasmuch as $y_n \in int\ K\left(\overline{\lambda}\right)$. From the C-lower hemicontinuity of $f(\cdot,\overline{x},\overline{\lambda})$ at y, we obtain that $f(y,\overline{x},\overline{\lambda}) \notin int\ C$.

As a result, $\overline{x} \in \Omega(\lambda)$, which implies that Ω is closed at λ.

Let us prove the second assertion. Suppose, to the contrary, that there exist an open set U of $\Omega\left(\overline{\lambda}\right)$ and a sequence $(\lambda_n\}$ converging to $\overline{\lambda}$ such that $x_n \in \Omega(\lambda_n) \setminus U$ for all $n \in N$. Then, one has

$$f(y, x_n, \lambda_n) \notin int\ C,\ \forall y \in K(\lambda_n).$$

By the compactness of K at $\overline{\lambda}$, one can assume without loss of generality that the sequence $\{x_n\}$ approaches some point in $K\left(\overline{\lambda}\right)$. An argument similar to the preceding part of the proof also shows that $\in \Omega\left(\overline{\lambda}\right)$, which admits a contradiction as $x_n \notin U$, for all n. Therefore, Ω is u.c. at $\overline{\lambda}$. This brings the proof to complete.

Regarding the upper continuity of solution maps to Stampacchia vector equilibrium problems, we obtain the following result. We would like to omit the proof because it is analogous to that of Theorem 3.8 under suitable adjustments.

Theorem 3.9. For the problem (PSEP), assume that
(i) K is a convex-compact-valued mapping satisfying $int K(\lambda) \neq \varnothing$ for all $\lambda \in \Lambda$;(ii) K is closed and l.c. on Λ;
(iii) For each $y \in X, f(\cdot,y,\cdot)$ is C-usc on $X \times \Lambda$.
Then, the solution map Ψ is closed on Λ. In addition, if K is compact on Λ, and for each $(x,\lambda) \in X \times \Lambda, f(x,\cdot,\lambda)$ is C-upper hemicontinuous on $K(\Lambda)$, then Ψ is u.c. on Λ.

The conclusions of Theorems 3.8 and 3.9 are still valid when we replace C-lower semicontinuity and C-upper semicontinuity assumptions on the objective maps by the closedness of $lev_{\not\succ 0_Y} f(y,\cdot,\cdot)$and $lev_{\not\prec 0_Y} f(\cdot,y,\cdot)$, respectively, on $X \times \Lambda$ for each $y \in X$.

LOWER STABILITY FOR MINTY AND STAMPACCHIA VECTOR EQUILIBRIUM PROBLEMS

This section aims to investigate sufficient conditions for the lower continuity of solution maps to Minty and Stampacchia vector equilibrium problems.

Theorem 4.1. *For the problem* (PMEP), *assume that*
(i) K is continuous and compact valued on Λ;
(ii) f is C-upper semicontinuous on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda), f(y,\cdot,\lambda)$ is strictly $(Y \setminus intC)$-quasiconcave on $K(\lambda)$.
If $\Omega(\lambda)$ has at least two elements for each $\lambda \in \Lambda$, then the solution map Ω is l.c. on Λ.

Proof. Suppose, to the contrary, that there is $\overline{\lambda} \in \Lambda$ such that Ω is not l.c. at $\overline{\lambda}$. Then, there exist $x_0 \in \Omega\left(\overline{\lambda}\right)$ and a neighborhood N_0 of the origin in $\mathscr{X}$ such that for all neighborhood U of $\overline{\lambda}$, there is $\lambda \in U$ with

$$\Omega(\lambda) \cap (x_0 + N_0) = \varnothing.$$

Consequently, there is a sequence $\{\lambda_n\}, \lambda_n \to \overline{\lambda}$ such that

$$\Omega(\lambda_n) \cap (x_0 + N_0) = \varnothing, \forall n \in N.$$

Because $\Omega\left(\overline{\lambda}\right)$ has at least two elements, we can choose $\overline{x} \in \Omega\left(\overline{\lambda}\right)$ with $\overline{x} \neq x_0$. As a result, for any $y \in K\left(\overline{\lambda}\right)$, we have

$$f\left(y, x_0, \overline{\lambda}\right) \in Y \setminus int\ C \text{ and } f(y, \overline{x}, \overline{\lambda}) \in Y \setminus int\ C.$$

By the strict $(Y \setminus int\ C)$-quasiconcavity of $f(y, \cdot, \overline{\lambda})$on $K\left(\overline{\lambda}\right)$, we get, $\forall t \in (0,1)$

$$f\left(y, t\overline{x} + (1-t)x_0, \overline{\lambda}\right) \in int\ (Y \setminus int\ C).$$

Since K is convex-valued at $\overline{\lambda}, x(t) \in K\left(\overline{\lambda}\right)$, where $x(t) = t\overline{x} + (1-t)x_0$. For the chosen N_0, there are a neighborhood N_1 of the origin in X and $t_0 \in (0,1)$ such that $N_1 + N_1 \subset N_0$ and $x(t_0) \in x_0 + N_1$. Consequently,

$$x(t_0) + N_1 \subset x(t_0) + N_1 + N_1 \subset x_0 + N_0.$$

Since K is l.c. at $\overline{\lambda}$, for $x(t_0) \in K\left(\overline{\lambda}\right)$ there exists $\overline{x}_n(t_0) \in K(\lambda_n)$ such that $\{\overline{x}_n(t_0)\}$ tends to $x(t_0)$, which yields that $\overline{x}_n(t_0) \in x(t_0) + N_1 \subset x_0 + N_0$, eventually. Thus, $\overline{x}_n(t_0) \notin \Omega(\lambda_n)$, for all n, which assures the existence of $\overline{y}_n \in K(\lambda_n)$ in order that

$$f(\overline{y}_n, \overline{x}_n(t_0),\ \lambda_n) \in\ int\ C.$$

By the upper continuity and compact-valuedness of K at $\overline{\lambda}$, one can assume without loss of generality that $\overline{y}_n$ converges to some point $\overline{y} \in K\left(\overline{\lambda}\right)$. For every neighborhood V of the origin in Y, there exists a balanced neighborhood V_1 of the origin in Y, i.e., $-V_1 = V_1$, such that $V_1 \subset V$. The C-upper semicontinuous of f ensures that

$$f(\overline{y}_n, \overline{x}_n(t_0), \lambda_n) \in\ f\left(\overline{y}, x(t_0), \overline{\lambda}\right) + V_1 - C,$$

or equivalently,

$$f\left(\overline{y}, x(t_0), \overline{\lambda}\right) \in f(\overline{y}_n, \overline{x}_n(t_0), \lambda_n) + V_1 + C,$$

because of the balance of V_1. From above and using the convexity of C, we have

$$f\left(\overline{y}, x(t_0), \overline{\lambda}\right) \in int\ C + V_1 + C \subset V + int\ C.$$

Since B is arbitrary and C is a closed cone, we have $f\left(\overline{y}, x(t_0), \overline{\lambda}\right) \in\ int\ C$, which is impossible. This contradiction brings the proof to the end.

Using this approach, we immediately obtain the lower stability of solution map to Stampacchia vector equilibrium problem.

Theorem 4.2. For the problem (PSEP), assume that
(i) K is continuous and compact valued on Λ;
(ii) f is C-lower semicontinuous on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda)$, $f(\cdot, y, \lambda)$ is strictly $(Y \setminus$-int $C)$-quasiconcave on $K(\lambda)$.
If $\Psi(\lambda)$ has at least two elements for each $\lambda \in \Lambda$, then the solution map Ψ is lsc on Λ.

Remark 4.3. In the special case mentioned in Remark 3.5, Theorem 4.2 is an improvement of Theorem 4.1 in [14], because it does not require any monotonicity properties on the objective map. Moreover, the convexity and continuity of the objective map are now relaxed. Kimura and Yao [23], and Ansari et al. [2] have established the lower semi-continuity of the solution maps to weak vector equilibrium problems by using strictly proper quasiconvexity property of the objective map (Theorem 5.1 in [23], and Theorem 9.37 in [2]). Theorem 4.2 slightly improves these theorems since the strictly proper quasiconvexity property is now relaxed by the strictly quasiconvexity one. Recently, Peng and Chang [24] have discussed the lower semi-continuity for weak efficient solution maps to parametric generalized systems under assumptions regarding monotonicity properties of objective map. Their approach (in Theorem 3.2 of [24]) based on the density property of the solution set, herein our approach is different from theirs. The following example is given to illustrate the mentioned case.

Example 4.4. Let $\mathscr{X} = T = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = [0,1], K(\lambda) = [\lambda, \lambda+2]$ and $f : X \times X \times \Lambda \to Y$ be defined as

$$f(x, y, \lambda) = \left(x^3 + y^3, x^2 - y^2\right).$$

It is obvious that all assumptions of Theorem 4.2 hold true, and so our Theorem 4.2 is applicable, and the solution map Ψ is lower continuous on Λ. Indeed, direct computations give us $\Psi(\lambda) = [\lambda, \lambda+2]$ for all $\lambda \in \Lambda$. However, monotonicity assumption (iii) of Theorem 3.2 in [24] is violated because $f(x,y,\lambda) + f(y,x,\lambda) = 2\left(x^3 + y^3, 0\right) \in C$ for all $x, y \in K(\lambda)$. As a result, Theorem 3.2 in [24] is not applicable in this case.

With the aim of relaxing the cone-semi-continuity properties on the objective maps of Minty and Stampacchia vector equilibrium problems, we have the following results.

Corollary 4.5. For the problem (PMEP), assume that
(i) K is continuous and compact valued on Λ;
(ii) $lev_{\not\leq 0_Y} f$ is closed on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda)$, $f(y, \cdot, \lambda)$ is strictly $(Y \setminus intC)$-quasiconcave on $K(\lambda)$.

If $\Omega(\lambda)$ has at least two elements for each $\lambda \in \Lambda$, then the solution map Ω is l.c. on Λ.

Corollary 4.6. For the problem (PSEP), assume that
(i) K is continuous and compact valued on Λ;
(ii) $lev_{\not\geq 0_Y} f$ is closed on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda), f(\cdot, y, \lambda)$ is strictly $(Y \setminus$-int $C)$-quasiconcave on $K(\lambda)$.
If $\Psi(\lambda)$ has at least two elements for each $\lambda \in \Lambda$, then the solution map Ψ is lsc on Λ.By combining results on the upper continuity and lower continuity of solution map to (PMEP), we obtain the following results regarding the continuity of solution map to the reference problem.

Theorem 4.7. For the problem (PMEP), assume that
(i) K is continuous and compact-convex-valued on Λ;
(ii) f is C-continuous on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda), f(y, \cdot, \lambda)$ is $(Y \setminus intC)$-quasiconcave on $K(\lambda)$.
Then, Ω is continuous on Λ.

Proof. According to Theorem 3.1, Ω is u.c. on Λ, hence it suffices to show that Ω is l.c. on Λ. Suppose, to the contrary, that there is $\overline{\lambda} \in \Lambda$ such that Ω is not l.c. at $\overline{\lambda}$. Then, we can find $x_0 \in \Omega\left(\overline{\lambda}\right)$ and a neighborhood W_0 of the origin in X such that for any neighborhood U of $\overline{\lambda}$, there is $\lambda \in U$ satisfying

$$\Omega(\lambda) \cap (x_0 + W_0) = \varnothing.$$

Thus, there is a sequence $\{\lambda_n\}, \lambda_n \to \overline{\lambda}$ such that

$$\Omega(\lambda_n) \cap (x_0 + W_0) = \varnothing, \forall n \in N$$

There are two situations needing to be considered.Case 1. $\Omega(\overline{\lambda}) = \{x_0\}$. Let $\{x_n\}$ be an arbitrary sequence in $\Omega(\lambda_n)$. By the upper continuity and compact-valuedness of K, we can assume that $\{x_n\}$ converges to some point $\overline{x}$ in $K\left(\overline{\lambda}\right)$ (taking a subsequence if necessary). Using the same arguments given in the proof of Theorem 3.1, we conclude that $\overline{x} \in \Omega\left(\overline{\lambda}\right)$, and hence $\overline{x} = x_0$. Consequently, $x_n \in x_0 + W_0$ eventually, a contradiction.
Case 2. $\Omega\left(\overline{\lambda}\right)$ is not a singleton. Proceed similarly as in the proof of Theorem 4.1, we also obtain another contradiction. These facts verify the claimed assertion and complete the proof of the theorem.
Similarly, we have a result related to the continuity of solution maps to Stampacchia vector equilibrium problems.

Theorem 4.8. For the problem (PSEP), assume that
(i) K is continuous and compact-convex-valued on Λ;
(ii) f is C-continuous on $X \times X \times \Lambda$;
(iii) for each $\lambda \in \Lambda$, and $y \in K(\lambda), f(., y, \lambda)$ is $(Y \setminus -int\ C)$-quasiconcave on $K(\lambda)$.
Then, Ψ is continuous on Λ.

Remark 4.9. Very recently, under the generalized convexity of objective maps Anh[18] has investigated the continuity of solution maps to parametric Stampacchia vector equilibrium problems with fixed constraints. However, this approach requires the solvability of an auxiliary problem whose solution set is defined as (Lemma 4.5 in[18]):

$$S_0^w(\lambda) = \{x \in X \mid f(x, y, \lambda) \nleq 0_Y, \forall y \in X\}.$$

Admittedly, this condition is very restrictive and cannot apply to some cases, for instance in the framework of variational inequalities. By imposing generalized convexity properties on objective maps, Theorems 4.7 and 4.8 provide sufficient conditions for the continuity of solution maps to the references problems without using the solvability of auxiliary problems. Therefore, our results are different from that of[18].

CONCLUSION

In this paper, upper and lower stability results for Minty and Stampacchia vector equilibrium problems were established under perturbations of the feasible set and objective mapping. By using cone-semicontinuity, generalized level closedness, and cone-convexity assumptions, the obtained results extend several existing works without requiring classical monotonicity conditions. These results provide a useful framework for further studies and applications in optimization and equilibrium theory.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

AUTHORS' CONTRIBUTIONS

All authors contributed to the discussion and writing of the final version equally.

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Bài nghiên cứu

Tính ổn định đối với các bài toán cân bằng véc-tơ

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TÓM TẮT

Bài báo này tập trung nghiên cứu tính ổn định của các bài toán cân bằng vectơ kiểu Minty và Stampacchia, đặc biệt trong các trường hợp mà cả tập chấp nhận được và ánh xạ mục tiêu đều chịu tác động của các nhiễu loạn. Nghiên cứu sử dụng các công cụ toán học hiện đại như tính nửa liên tục theo nón và mức độ đóng suy rộng của các ánh xạ mục tiêu để thiết lập các kết quả quan trọng về ổn định trên đối với các bài toán cân bằng vectơ này. Bên cạnh đó, bài báo cũng khảo sát các tính chất ổn định dưới thông qua việc áp dụng các giả thiết lồi theo nón suy rộng trên các ánh xạ mục tiêu, đáng chú ý là không cần sử dụng các tính chất đơn điệu truyền thống vốn thường khá hạn chế trong các ứng dụng thực tiễn.

Các kết quả đạt được trong bài báo đóng góp đáng kể cho lĩnh vực bài toán cân bằng vectơ bằng cách mở rộng phạm vi của các lý thuyết ổn định hiện có. Công trình này khắc phục một số hạn chế của các nghiên cứu trước đây, vốn thường yêu cầu tính liên tục chặt chẽ của ánh xạ mục tiêu hoặc điều kiện tồn tại nghiệm của các bài toán phụ trợ. Việc nới lỏng các giả thiết này cho phép xây dựng một khuôn khổ phân tích ổn định mạnh hơn và linh hoạt hơn đối với các bài toán cân bằng vectơ, từ đó giúp các kết quả có thể áp dụng cho nhiều tình huống thực tiễn hơn, bao gồm các lĩnh vực vật lý, kỹ thuật, kinh tế học và phân tích mạng xã hội.

Cuối cùng, bài báo này góp phần nâng cao hiểu biết về tính ổn định của các bài toán cân bằng vectơ, đồng thời tạo nên tảng cho các nghiên cứu tiếp theo cũng như các ứng dụng tiềm năng trong nhiều lĩnh vực khoa học và kỹ thuật khác nhau. Những phát triển lý thuyết được trình bày trong bài không chỉ cải thiện việc mô hình hóa toán học các hệ thống phức tạp mà còn góp phần thúc đẩy việc triển khai thực tiễn các mô hình này trong các bài toán thực tế. Bên cạnh đó, một số ví dụ minh họa cũng được đưa ra nhằm chứng minh tính khả thi và hiệu quả của các kết quả ổn định đạt được. Các ví dụ này làm nổi bật tính linh hoạt của phương pháp được đề xuất và nhấn mạnh tiềm năng ứng dụng của nó trong việc giải quyết các mô hình cân bằng phức tạp phát sinh trong tối ưu hóa, khoa học ra quyết định và phân tích đa tiêu chí.

Từ khoá: Bài toán cân bằng vectơ, Ổn định trên, Ổn định dưới, Tính liên tục theo nón, Tính lồi theo nón

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