

Convergence of the solutions for weak vector quasi-equilibrium problems

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ABSTRACT

In this paper, we focus on the study of weak vector quasi-equilibrium problems (WVQEPs) in finite-dimensional vector spaces, which generalize various equilibrium models in vector optimization and game theory. We begin by formulating the weak vector quasi-equilibrium problems and establishing the foundational assumptions that ensure the well-posedness of such models. Our approach integrates set-valued analysis and variational techniques to handle the vector-valued nature of the problem.

Next, we introduce the notions of auxiliary solution sets associated with the WVQEPs. These sets play a crucial role in analyzing the qualitative behavior of solution mappings. In particular, we define and explore a quasi-convexity condition on the objective functions, which generalizes classical convexity and enables us to treat a broader class of problems where convexity may not hold. This quasi-convexity condition is shown to be sufficient for ensuring certain continuity and convergence properties of the solution sets.

Building on these developments, we investigate the Painlevé-Kuratowski lower convergence of the solution sets corresponding to a sequence of WVQEPs defined by a sequence of continuous mappings. We establish conditions under which the solution sets exhibit lower semi-continuity and set convergence, and we provide a rigorous mathematical framework for analyzing stability properties of the equilibrium solutions under perturbations. These results contribute to the understanding of how solution sets behave as the underlying functions change continuously.

To illustrate the theoretical findings, several examples are constructed, demonstrating the applicability of our assumptions and the sharpness of our conclusions. The examples include cases that are not covered by existing convexity-based results, emphasizing the novelty and generality of our approach.

Our main results extend and differ from several well-known findings in the literature on vector equilibrium problems. In particular, the use of quasi-convexity and auxiliary solution sets allows us to bypass some restrictive conditions found in previous works, offering a new perspective on the stability analysis of vector equilibrium problems. These contributions enrich the theory of vector optimization and open new directions for future research in variational analysis and generalized equilibria.

Key words: Weak quasi-equilibrium problems, Painlevé-Kuratowski upper (lower) convergence, Painlevé-Kuratowski convergence

INTRODUCTION

Let $\mathcal{X}, \mathcal{Z}$ be two Banach spaces, $X \subset \mathcal{X}$; $Z \subset \mathcal{Z}$ are nonempty, compact, convex subsets. Let $H: X \rightrightarrows X$; $Q: X \rightrightarrows Z$ be set-valued mappings and $h: X \times Z \times X \rightarrow \mathbb{R}^P$ be a vector function.

Denote the nonnegative of $\mathbb{R}_+^P$ and its interior, respectively, by

$$\mathbb{R}_+^P := \left\{ t = (t_1, t_2, \dots, t_p) \in \mathbb{R}^P \mid t_i \geq 0, i = 1, 2, \dots, p \right\} \text{ and}$$

$$\text{int } \mathbb{R}_+^P := \left\{ t = (t_1, t_2, \dots, t_p) \in \mathbb{R}^P \mid t_i > 0, i = 1, 2, \dots, p \right\}.$$

We consider the following *weak vector quasi-equilibrium problems*.

(WQVEP): Find $x \in H(x)$ such that

$$\forall z \in Q(x) \mid h(x, z, y) \notin -\text{int } \mathbb{R}_+^P, \forall y \in H(x). \quad (1.1)$$

For sequences of set-valued mappings $H_n: X \rightrightarrows X$ and $Q_n: X \rightrightarrows Z$ and single-valued mappings $h_n: X \times Z \times X \rightarrow \mathbb{R}^P$, for $n \in \mathbb{N} \setminus \{0\}$. We consider the following *sequence of weak vector*

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quasi-equilibrium problems.

(WQVEP) : Find $x \in H_n(x)$ such that

$$\forall z \in Q_n(x) \mid h_n(x, z, y) \notin -\text{int } \mathbb{R}^p, \forall y \in H_n(x). \quad (1.2)$$

We denote the solution sets of problems (WQVEP) and $(\text{WQVEP})_n$ by $\Theta(h, Q, H)$ and $\Theta(h_n, Q_n, H_n)$, respectively. Throughout this paper, we always assume that $\Theta(h, Q, H)$ and $\Theta(h_n, Q_n, H_n)$ are nonempty sets.

As we know that the concept of Painlevé-Kuratowski (P.K.) convergence of solution sets for optimization-related problems has an important role in optimization theory and applications. There are many papers on the Painlevé-Kuratowski convergence of solutions, for example optimization problems, see [4,6,8], multiobjective games, see [7], and equilibrium problems, see [1,2,9]). Very recently, Duy et al. [3] established strong vector quasi-equilibrium problems and sequences of strong vector quasi-equilibrium problems of the Minty type and the Stampacchia type. Then, they introduced the auxiliary subsets of the solution sets for these problems. The concept of generalized C -quasi-concavity of the objective functions for strong vector quasi-equilibrium problems is studied. Finally, they investigated sufficient conditions for the lower convergence, upper convergence and convergence in the Painlevé-Kuratowski sense with sequences of mappings converging continuously to strong vector quasi-equilibrium problems. However, to the best of our knowledge, there are not any papers on the convergence results by using the auxiliary solution sets and the concept of $\mathbb{R}_+^p$ -quasi-convexity of the objective functions for weak vector quasi-equilibrium problems. That is why we establish the convergence of the solution sets for weak vector quasi-equilibrium problems.

In the following, we recall some concepts of the convergence of set sequences and mapping sequences [2, 10].

Let X be a normed space. A sequence of sets $\{D_n\}$, $D_n \subseteq X$, is said to *upper converge* (*lower converge*, respectively) in the sense of Painlevé-Kuratowski to D , if $\limsup_{n \rightarrow \infty} D_n \subseteq D$ ($D \subseteq \liminf_{n \rightarrow \infty} D_n$, respectively). $\{D_n\}$ is said to *converge* in the sense of Painlevé-Kuratowski to D , if $\limsup_{n \rightarrow \infty} D_n \subseteq D \subseteq \liminf_{n \rightarrow \infty} D_n$ with

$$\liminf_{n \rightarrow \infty} D_n := \left\{ x \in X : x = \lim_{n \rightarrow \infty} x_n, x_n \in D_n, \text{ for sufficiently large } n \right\}$$

$$\limsup_{n \rightarrow \infty} D_n := \left\{ x \in X : x = \lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \in D_{n_k}, \{x_{n_k}\} \text{ a subsequence of } \{x_n\} \right\}$$

Let $H_n: X \rightrightarrows Y$ be a sequence of set-valued mappings and $H: X \rightrightarrows Y$ be a set-valued mapping. $\{H_n\}$ is said to *outer converge continuously* (*inner converge continuously*, respectively) to H at x_0 if $\limsup_{n \rightarrow \infty} H_n(x_n) \subseteq H(x_0)$ ($H(x_0) \subseteq \liminf_{n \rightarrow \infty} H_n(x_n)$, respectively) $\forall x_n \rightarrow x_0$, $\{H_n\}$ is said to *converge continuously* to H at x_0 if $\limsup_{n \rightarrow \infty} H_n(x_n) \subseteq H(x_0) \subseteq \liminf_{n \rightarrow \infty} H_n(x_n) \forall x_n \rightarrow x_0$. If $\{H_n\}$ converges continuously to H at every $x_0 \in X$, then it is said that $\{H_n\}$ converges continuously to H in X .

Now, we recall a concept of the continuous convergence for a sequence of single-valued mappings. A sequence of mappings $\{h_n\}$, where $h_n: X \rightarrow Y$, is said to *converge continuously* to a mapping $h: X \rightarrow Y$ at x_0 if $\lim_{n \rightarrow \infty} h_n(x_n) = h(x_0), \forall x_n \rightarrow x_0$. If $\{h_n\}$ converges continuously to h at every $x_0 \in X$, then it is said that $\{h_n\}$ converges continuously to h in X .

MAIN RESULTS

Firstly, we will study sufficient conditions of lower convergence of the solution sets for (WQVEP) and $(\text{WQVEP})_n$.

For all $n \in \mathbb{N}$, we consider the following auxiliary subsets of $\Theta(h, Q, H)$ and $\Theta(h_n, Q_n, H_n)$, respectively.

$$\Theta(h, Q, H) = \{x \in H(x) : \forall z \in Q(x) \mid h(x, z, y) \notin -\text{cl}\mathbb{R}_+^p, \forall y \in H(x)\} \text{ and}$$

$\Theta(h_n, Q_n, H_n) = \{x \in H_n(x) : \forall z \in Q_n(x) \mid h_n(x, z, y) \notin -\text{cl}\mathbb{R}_+^p, \forall y \in H_n(x)\}$, where $\text{cl}\mathbb{R}^p$ is the closure of $\mathbb{R}^p$. It is clear that

$$\Theta(h, Q, H) \subseteq \Theta(h, Q, H) \text{ and } \Theta(h_n, Q_n, H_n) \subseteq \Theta(h_n, Q_n, H_n).$$

We also always assume that $\Theta(h, Q, H) \neq \emptyset$ and $\Theta(h_n, Q_n, H_n) \neq \emptyset$.

We now will introduce the concept of $\mathbb{R}_+^p$ -quasi-concavity for a sequence of weak vector quasi-equilibrium problems.

Definition 2.1 Let X be a Banach space and $h_n: X \times X \rightarrow \mathbb{R}^p$ be a sequence of vector functions. Then, h_n is said to be $\mathbb{R}_+^p$ -quasi-concave in $X \times X$ if for each $x_1, x_2 \in X$ and $y_1, y_2 \in X$ such that $h_n(x_1; y_1) \notin -\text{int}\mathbb{R}_+^p$ and $h_n(x_2; y_2) \notin -\text{cl}\mathbb{R}_+^p, \forall \alpha \in (0; 1)$, we have

$$h_n(\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2) \notin -\text{cl}\mathbb{R}_+^p, \forall n \in \mathbb{N}.$$

Theorem 2.1 Suppose that the problems (WQVEP) and $(\text{WQVEP})_n$ satisfies the following conditions:

- (i) H_n converges continuously to H with compact values and concave in X ;
- (ii) Q_n outer converges continuously to Q with compact values in X ;
- (iii) $h_n(\cdot, z, \cdot)$ is $\mathbb{R}_+^p$ -quasi-concave in $X \times X$, for all $z \in Z$;
- (iv) h_n converges continuously to h .

Then $\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$.

Proof. We show that $\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$.

Assume to the contrary that $\Theta(h, Q, H) \not\subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$. Then, there exists

$x_0 \in \Theta(h, Q, H)$ for all $x_n \in \Theta(h_n, Q_n, H_n)$ such that x_n does not converge to x_0 . From H_n inner

converges continuously to H , for all $x_0 \in H(x_0)$, there exists a sequence $\{\hat{x}_n\} \subseteq H_n(\hat{x}_n)$ such that $\{\hat{x}_n\}$ converges to x_0 . By the above contradiction assumption, there must be a subsequence $\{\hat{x}_k\}$ of $\{\hat{x}_n\}$ such that, $\forall k, \hat{x}_k \notin \Theta(h_k, Q_k, H_k)$, that is, there exist $z_k \in Q_k(\hat{x}_k)$ and $y_k \in H_k(\hat{x}_k)$ such that

$$h_k(\hat{x}_k, z_k, y_k) \in -\text{cl}\mathbb{R}_+^p. \quad (2.1)$$

Since H_n outer converges continuously to H with compact values in X , there is $y_0 \in H(x_0)$ such that $y_k \rightarrow y_0$, also Q_n outer converges continuously to Q with

compact values in X , there is $z_0 \in Q(x_0)$ such that $z_k \rightarrow z_0$. From h_n converges continuously to h , $(\hat{x}_k, z_k, y_k) \rightarrow (x_0, z_0, y_0)$ and (2.1), we have

$$h(x_0, z_0, y_0) \in -\text{cl}\mathbb{R}_+^p,$$

which is impossible, since $x_0 \in \Theta(h, Q, H)$. Thus,

$$\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n).$$

Now let us check that $\Theta(h, Q, H) \subseteq \text{cl}\Theta(h, Q, H)$. Indeed, let $x_1 \in \Theta(h, Q, H)$, $x_2 \in \Theta(h, Q, H)$ and $x_\alpha = \alpha x_1 + (1-\alpha)x_2, \alpha \in (0,1)$. By the convexity of X , we have $x_\alpha \in X$. By H_n is concave, for all $y_\alpha \in H_n(x_\alpha)$, there exist $y_1 \in H_n(x_1)$, $y_2 \in H_n(x_2)$ such that $y_\alpha = \alpha y_1 + (1-\alpha)y_2$. From the condition $h_n(., z, .)$ is $\mathbb{R}_+^p$ -quasi-concave in $X \times X$, for all $z \in Z$, we have

$$h_n(x_\alpha, z, y_\alpha) \notin -\text{cl}\mathbb{R}_+^p,$$

i.e., $x_\alpha \in \Theta(h, Q, H)$. Hence $\Theta(h, Q, H) \subseteq \text{cl}\Theta(h, Q, H)$.

From $\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$, we have

$$\Theta(h, Q, H) \subseteq \text{cl}\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n).$$

Therefore, we have,

$$\Theta(h, Q, H) \subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n).$$

Thus, the proof is completed. $\square$

The following example proves that the assumption H_n converges continuously to H of Theorem 2.1 is essential.

Example 2.1 Let $\mathcal{X} = \mathcal{Z} = \mathbb{R}$, $X = Z = [-1,1]$, $p = 1$, $H, H_n : [-1,1] \rightrightarrows [-1,1]$, $Q, Q_n : [-1,1] \rightrightarrows [-1,1]$ and $h, h_n : [-1,1] \times [-1,1] \times [-1,1] \rightarrow \mathbb{R}_+$ be defined by

$$H(x) = [-1,1], H_n(x) = \left[-1 - \frac{1}{n+1}, \frac{1}{n+1}\right],$$

$$Q(x) = Q_n(x) = \left\{\frac{1}{e^x}\right\},$$

$$h(x, z, y) = z + zx^4,$$

$$h_n(x, z, y) = \left(z + \frac{1}{n+1}\right)(1 + x^4).$$

It is easy to get after a direct computation $\Theta(h, Q, H) = [-1,1]$ and

$\Theta(h_n, Q_n, H_n) = \left[-1 - \frac{1}{n+1}, \frac{1}{n+1}\right]$. It is clear that all the conditions of Theorem 2.1 are

satisfied except the condition H_n converges continuously to H . The reason is that H_n does not

inner converge continuously to H (H_n only outer converges continuously to H). Hence, the result of Theorem 2.1 not hold. In fact,

$$[-1, 1] = \Theta(h, Q, H) \not\subseteq \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) = [-1, 0].$$

The following example illustrates that the condition Q_n outer converges continuously to Q of Theorem 2.1 is essential.

Example 2.2 Let $\mathcal{X} = \mathcal{Z} = \mathbb{R}$, $X = Z = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $p = 1$, $H, H_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right]$, $Q, Q_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right]$, and $h, h_n : \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \rightarrow \mathbb{R}_+$ be defined by

$$\begin{aligned} H(x) &= H_n(x) = \left[0, \frac{1}{2}\right], \\ Q(x) &= \left\{0, \frac{1}{2}\right\}, Q_n(x) = \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}, \\ h(x, z, y) &= x + z + y, \\ h_n(x, z, y) &= \left(1 + \frac{1}{n+1}\right)(x + z + y). \end{aligned}$$

It follows from a direct computation $\Theta(h, Q, H) = \left[0, \frac{1}{2}\right]$ and $\Theta(h_n, Q_n, H_n) = \left\{\frac{1}{2}\right\}$. We see that all the conditions of Theorem 2.1 are satisfied except the condition Q_n outer converges continuously to Q . Thus, the result of Theorem 2.1 not hold. In fact,

$$\left\{\frac{1}{2}\right\} = \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) \not\subseteq \Theta(h, Q, H) = \left[0, \frac{1}{2}\right].$$

The following example demonstrates that the $\mathbb{R}_+^p$ -quasi-concavity of h_n of Theorem 2.1 is essential.

Example 2.3 Let $\mathcal{X} = \mathcal{Z} = \mathbb{R}$, $X = Z = [-2, 2]$, $p = 1$, $H, H_n : [-2, 2] \Rightarrow [-2, 2]$, $Q, Q_n : [-2, 2] \Rightarrow [-2, 2]$ and $h, h_n : [-2, 2] \times [-2, 2] \times [-2, 2] \rightarrow \mathbb{R}_+$ be defined by

$$\begin{aligned} H(x) &= [0, 2], H_n(x) = \left[\frac{1}{n+1}, \frac{1}{n+1} + 2\right], \\ Q(x) &= Q_n(x) = [0, 1], h(x, z, y) = x^2 - x, \\ h_n(x, z, y) &= \left(1 + \frac{1}{n+1}\right)(x^2 - x). \end{aligned}$$

It is obtained after a direct calculation

$$\Theta(h, Q, H) = \{0\} \cup [1, 2]$$

and

$$\Theta(h_n, Q_n, H_n) = \left[1 + \frac{1}{n+1}, \frac{1}{n+1} + 2\right].$$

We see that all the assumptions of Theorem 2.1 are satisfied except the $\mathbb{R}_+^p$ -quasi-concavity of h_n . Indeed, take $x_1 = 0, x_2 = \frac{3}{2} \in H(0) = [0, 2]$, then for all $y \in H(0) = [0, 2]$, for all $z \in Z$, we have $h_n(x_1, z, y) = 0 \in [0, +\infty), h_n(x_2, z, y) = \frac{3}{4} \in (0, +\infty)$ but $h_n\left(\frac{1}{2}x_1 + \frac{1}{2}x_2, z, y\right) = -\frac{3}{16}\left(1 + \frac{1}{n+1}\right) \notin (0, +\infty)$. Hence, the result of Theorem 2.1 not hold. In fact,

$$[1, 2] = \liminf_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) \not\subseteq \Theta(h, Q, H) = \{0\} \cup [1, 2].$$

Next, we will study the sufficient conditions of convergence of the solution sets for (WQVEP) and (WQVEP)_n.

Theorem 2.2 Assume that the problems (WQVEP) and (WQVEP)_n satisfies the following conditions:

- (i) H_n converges continuously to H with compact values and concave in X ;
 - (ii) Q_n converges continuously to Q with compact values in X ;
 - (iii) $h_n(\cdot, z, \cdot)$ is $\mathbb{R}_+^p$ -quasi-concave in $X \times X$ for all $z \in Z$;
 - (iv) h_n converges continuously to h .
- Then

$$\Theta(h_n, Q_n, H_n) \xrightarrow{p.k.} \Theta(h, Q, H), \text{ as } n \rightarrow \infty.$$

Proof. From the proof of Theorem 2.1, we only need to prove that

$$\limsup_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) \subseteq \Theta(h, Q, H).$$

Indeed, assume to the contrary that $\limsup_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) \not\subseteq \Theta(h, Q, H)$, i.e., there exists $x_0 \in \limsup_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$, but $x_0 \notin \Theta(h, Q, H)$. From $x_0 \in \limsup_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n)$, there exists a sequence $\{x_{n_k}\} \subseteq \Theta(h_{n_k}, Q_{n_k}, H_{n_k})$, $\{x_{n_k}\}$ converges to x_0 such that for all $z \in Q_{n_k}(x_{n_k})$, we have

$$h_{n_k}(x_{n_k}, z, y) \notin -\text{int } \mathbb{R}_+^p, \forall y \in H_{n_k}(x_{n_k}).$$

By H_{n_k} outer converges continuously to H and H_{n_k} is closed in X , we have $x_0 \in H(x_0)$. Now we prove that $x_0 \in \Theta(h, Q, H)$. If $x_0 \notin \Theta(h, Q, H)$, there exist $z_0 \in Q(x_0), y_0 \in H(x_0)$ such that

$$h(x_0, z_0, y_0) \in -\text{int } \mathbb{R}_+^p. \quad (2.2)$$

Since H_{n_k} inner converges continuously to H and Q_{n_k} inner converges continuously to Q , for all $z_0 \in Q(x_0)$ and $y_0 \in H(x_0)$ there exist $z_{n_k} \in Q_{n_k}(x_{n_k})$ and $y_{n_k} \in H_{n_k}(x_{n_k})$ such that $z_{n_k} \rightarrow z_0$ and $y_{n_k} \rightarrow y_0$ as $k \rightarrow \infty$. Since $x_{n_k} \in \Theta(h_{n_k}, Q_{n_k}, H_{n_k})$ we have

$$h_{n_k}(x_{n_k}, z_{n_k}, y_{n_k}) \notin -\text{int } \mathbb{R}_+^p. \quad (2.3)$$

Since h_{n_k} converges continuously to h , $(x_{n_k}, z_{n_k}, y_{n_k}) \rightarrow (x_0, z_0, y_0)$ and from (2.3), we have

$$h(x_0, z_0, y_0) \notin -\text{int } \mathbb{R}_+^p.$$

It is in contradiction with (2.2), and so the proof is completed. $\square$

The following example shows that all assumptions of Theorem 2.2 are fulfilled.

Example 2.4 Let $\mathcal{X} = \mathcal{Z} = \mathbb{R}$, $X = Z = [-3, 3]$, $p = 1$, $H, H_n : [-3, 3] \rightrightarrows [-3, 3]$, $Q, Q_n : [-3, 3] \rightrightarrows [-3, 3]$ and $h, h_n : [-3, 3] \times [-3, 3] \times [-3, 3] \rightarrow \mathbb{R}_+$ be defined by

$$\begin{aligned} Q(x) &= Q_n(x) = \{3^x\}, \\ H(x) &= [-1, 2], H_n(x) = \left[-1 - \frac{2}{n+1}, 2 + \frac{1}{n+1}\right], \\ h(x, z, y) &= z(4 - 3x - x^2), \\ h_n(x, z, y) &= \left(z + \frac{1}{n}\right)(4 - 3x - x^2). \end{aligned}$$

It follows from a direct computation

$$\Theta(h, Q, H) = [-1, 1]$$

and

$$\Theta(h_n, Q_n, H_n) = [-4, 1] \cap \left[-1 - \frac{2}{n+1}, 2 + \frac{1}{n+1}\right].$$

We show that all the assumptions of Theorem 2.2 are fulfilled. Then, the result of Theorem 2.2 hold, and so

$$\lim_{n \rightarrow \infty} \Theta(h_n, Q_n, H_n) = [-1, 1] = \Theta(h, Q, H).$$

Remark 2.1 As mentioned in the introduction, up to now, there are not any papers on the convergence results by using the auxiliary solution sets and the concept of $\mathbb{R}_+^p$ -quasi-convexity of the objective functions for weak vector quasi-equilibrium problems. Hence, our results in this section are new and different some main results in the literature.

CONCLUSION

In this paper, we have studied the convergence behavior of solution sets for weak vector quasi-equilibrium problems in finite-dimensional spaces. By introducing auxiliary solution sets and employing the concept of C-quasi-convexity (or quasi-concavity) of the objective functions, we established sufficient conditions for the Painlevé-Kuratowski lower convergence and full convergence of these solution sets under sequences of mappings that converge continuously.

Our results generalize and complement existing convergence theorems in the literature by removing some restrictive assumptions, such as convexity of the objective function, and replacing them with quasi-convexity-type conditions. Through a series of carefully constructed

examples, we have demonstrated the necessity of each condition in our theorems and illustrated the applicability of the proposed framework.

To the best of our knowledge, this is the first time auxiliary solution sets and the notion of quasi-convexity have been used in the context of weak vector quasi-equilibrium problems to study convergence. Hence, the theoretical findings in this paper provide new insights and open up promising directions for further research on the stability and sensitivity analysis of vector equilibrium problems and their generalizations.

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ABBREVIATIONS

WQVEP: weak vector quasi-equilibrium problems

(WQVEP): sequence of weak vector quasi-equilibrium problems

P.K.: Painlevé Kuratowski

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

AUTHOR CONTRIBUTIONS

Nguyen Xuan My: Conceptualization, formal analysis, writing—original draft.

Le Xuan Dai: Methodology, validation, supervision, writing—review & editing.

Nguyen Hong Phuc: Investigation, visualization, examples and counterexamples.

Nguyen Van Hung: Literature review, theoretical framework, writing—editing support.

All authors have read and approved the final manuscript.

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Sự hội tụ của tập nghiệm đối với bài toán tựa cân bằng vectơ yếu

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TÓM TẮT

Trong bài báo này, chúng tôi tập trung nghiên cứu các bài toán tựa cân bằng vectơ yếu (WVQEPs) trong không gian vectơ hữu hạn chiều, một dạng tổng quát của nhiều mô hình cân bằng trong tối ưu hóa vectơ và lý thuyết trò chơi. Trước tiên, chúng tôi xây dựng công thức bài toán tựa cân bằng vectơ yếu và thiết lập các giả thiết nền tảng nhằm đảm bảo bài toán được xác định tốt. Phương pháp tiếp cận của chúng tôi kết hợp giữa giải tích đa trị và kỹ thuật biến phân để xử lý bản chất vectơ của bài toán.

Tiếp theo, chúng tôi giới thiệu khái niệm về các tập nghiệm phụ trợ liên quan đến các bài toán WVQEP. Những tập này đóng vai trò quan trọng trong việc phân tích hành vi định tính của ánh xạ nghiệm. Đặc biệt, chúng tôi định nghĩa và khảo sát điều kiện lỗi xấp xỉ (quasi-convexity) đối với hàm mục tiêu, như một sự mở rộng của tính lỗi cổ điển, từ đó cho phép xử lý một lớp bài toán rộng hơn trong đó tính lỗi có thể không được đảm bảo. Điều kiện lỗi xấp xỉ này được chứng minh là đủ để đảm bảo một số tính chất liên tục và hội tụ nhất định của tập nghiệm.

Dựa trên những phát triển này, chúng tôi nghiên cứu sự hội tụ dưới theo nghĩa Painlevé–Kuratowski của các tập nghiệm ứng với một dãy bài toán WVQEP được xác định bởi một dãy ánh xạ liên tục. Chúng tôi thiết lập các điều kiện để các tập nghiệm thể hiện tính nửa liên tục dưới và hội tụ tập hợp, đồng thời cung cấp một khung toán học chặt chẽ nhằm phân tích tính ổn định của nghiệm cân bằng dưới các nhiễu loạn. Các kết quả này góp phần làm sáng tỏ hành vi của tập nghiệm khi các ánh xạ nền thay đổi liên tục.

Để minh họa cho các kết quả lý thuyết, chúng tôi xây dựng một số ví dụ cụ thể, cho thấy tính khả thi của các giả thiết được đề xuất và tính chính xác của các kết luận. Các ví dụ bao gồm những trường hợp không nằm trong phạm vi áp dụng của các kết quả dựa trên tính lỗi hiện có, từ đó nhấn mạnh tính mới và tính khái quát trong cách tiếp cận của chúng tôi.

Các kết quả chính của chúng tôi không chỉ mở rộng mà còn khác biệt so với nhiều kết quả nổi bật đã công bố trong tài liệu về bài toán cân bằng vectơ. Cụ thể, việc sử dụng điều kiện lỗi xấp xỉ và các tập nghiệm phụ trợ cho phép loại bỏ một số giả thiết hạn chế trong các công trình trước đó, đồng thời mở ra một góc nhìn mới về phân tích ổn định cho các bài toán cân bằng vectơ. Những đóng góp này làm phong phú thêm lý thuyết tối ưu vectơ và mở rộng các hướng nghiên cứu trong phân tích biến phân và lý thuyết cân bằng tổng quát.

Từ khóa: Bài toán tựa cân bằng yếu, hội tụ trên (dưới) Painlevé–Kuratowski, hội tụ Painlevé–Kuratowski.

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