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Research Article

DIRECT METHOD FOR ANALYSIS OF PRESSTRESSED CONCRETE STRUCTURE

Phạm Thành Công[1,*], Vũ Việt Hoàng[2]

ABSTRACT

One of significant progress of construction industry is prestressed concrete. The range of applications is very wide in the high-rise buildings, large-span buildings and the bridges. Prestressed concrete has been successfully used for small as well as large projects over the last sixty years. The efficiency stems from being able to use high strength materials, to structurally utilize the entire cross section, to vary the force and location of the reinforcing to best resist applied loads, and to control the timing of when the prestressing force is applied to the structure. In prestressed concrete structures, load balancing is an indirect and familiar method for analyzing prestressed members through equivalent load. The frame being subjected to equivalent load produces balanced moment or prestressing moment (it is taken into account in serviceability limit state) and reactions; then, secondary moment (it is taken into account in ultimate limit state) is produced by reactions. The paper presents a direct method for analyzing prestressed members by solving the differential equations of the elastic line. These differential equations consider the equilibrium of the reactions and eccentricity of the prestressing force. After solving these equations, the engineers can directly obtain the reactions; and subsequently, determine the prestressing moment, secondary moment and shear force. The direct method also finds out the precise effect resulting from large tendon sag in transfer or deep beam. Effect of large tendon sag is the same as small tendon sag, the differences are very small. Besides, in this paper, the equivalent load models associated with any complicated tendon profiles which are formularized deal with and expand the unconventional tendon profile in the future of construction industry.

Key words: differential equation of the elastic line, large tendon sag, prestressing moment, secondary moment, reactions due to prestressing force, prestressed concrete structure, structural analysis

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INTRODUCTION

Prestressed concrete is a highly efficient structural system that offers many benefits in a wide range of construction, repair, and rehabilitation applications. Prestressed concrete has been successfully used for small as well as large projects over the last sixty years. The efficiency stems from being able to use high strength materials, to structurally utilize the entire cross section, to vary the force and location of the reinforcing to best resist applied loads, and to control the timing of when the prestressing force is applied to the structure. Prestressed concrete offers a perfect balance of two materials which complement each other. Concrete is strong in compression and relatively weak in tension. The tensile strength of concrete is about 10% of its compressive strength. Prestressing steel, on the other hand, has a very high tensile strength which is about four times that of common reinforcing bars. By combining the two, a structural member can resist both compressive and tensile forces caused by various loads. This results in greater efficiency in resisting tensile as well as compressive stresses resulting from the applied loads. Prestressed concrete can be used in all facets of construction from buildings and bridges to highway pavements, slabs-on-ground and ground anchors. It has also been used for rehabilitation and retrofit applications.

In prestressed concrete structures, prestressing force induces two effects that are the axial prestress and eccentricity prestress; and to take eccentricity into account, load balancing was introduced by T. Y. Lin[1] or Moorman[2] (Moorman called equivalent load) as a simple yet powerful alternative method for analyzing prestressed members, it has been unique to calculating the effects of prestressing force on members for 60 years. In this conventional method, B. O. Alami[3] further presented, the tendon in Figure 1(a) is removed from its duct and replaced with the forces the tendon exerts on the structure when in place as well as a constant compression force. The replaced forces are comprised of upward and downward forces resulting from the tendon profile, called equivalent load, or balanced load. The frame being subjected to equivalent

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load produces balanced moment[3] or prestressing moment[4,5] (it is taken into account in serviceability limit state); then, secondary moment[4,5] (it is taken into account in ultimate limit state); and support reactions are calculated directly or indirectly[6] through prestressing moment. First, the effects of axial loading and the flexure due to prestressing force are independent from one another; therefore, the general behavior must be considered individually[6]. Second, this method must be accepted that the eccentricity of the prestressing force is small in comparison to the span[7,8] and the studies about prestressing force is large in comparison to the span, on the other hand the large tendon sag has not investigated adequately or referred to in documentaries, standards. Third, for the tendons have cubic or any complicated profiles, a basic theory about equivalent load need to make clear in this cases. Fourth, load balancing method does not consider the effect of frictional loss[7]. In reality, tendon force profile varies along the length of the tendon due to the frictional loss[9−11] and in order to overcome this an average applied force (approximate force) on the member is replaced. This is the limit of the load balancing method.
The paper presents a direct method to determine the prestressing, secondary moment and shear force[12] by solving the differential equations of the elastic line. Through this method, the nature of the prestressing force in member which includes the tensile force in tendons and its eccentricity is retained. Thus, the engineers can directly consider the eccentricity of the tendons, the tensile force, and the reactions in the equilibrium equations without using equivalent load or removing the tendons. After solving these equations, the engineers can directly obtain the reactions; and subsequently, determine the prestressing, secondary moment and shear force.
Besides, in this paper, the bending moment derives from reactions and prestressing force in the equilibrium. And based on considering the relationship between second derivatives of this bending moment with respect to distance, the equivalent load models associated with any complicated tendon profiles which are formularized deal with and expand the unconventional tendon profile in the future of construction industry.

EQUILIBRIUM OF PRESTRESSED MEMBER AND DIFFERENTIAL EQUATION OF THE ELASTIC LINE

Considering any prestressed member with a small tendon sag, the assumption that after prestressing, the member is deformed according to the function y = f(x), called deformation line in Figure 1(a). The supports constrain this deformation of the member, and reactions consist of the forces and moments shown in Figure 1(a) are generated.

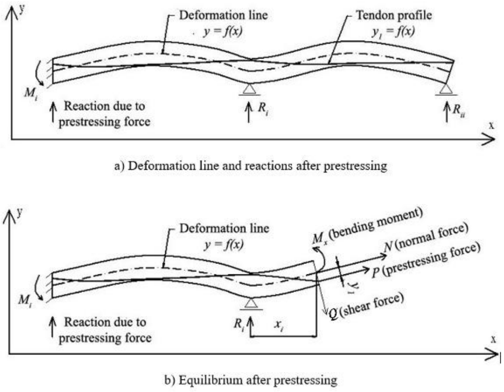


Figure 1. The deformation line, reactions and equilibrium after prestressing [Source: The Authors]

Passing a cut section at location x, the equilibrium in Figure 1(b) (deformation due to shear force is ignored) can be expressed as:

$$\Sigma M_i = M_x + M_1 - \Sigma R_i x_i + P|y_1| = 0 \qquad (1)$$
$$\Rightarrow M_x = -M_1 + \Sigma R_i x_i + P y_1$$

The differential equation of the elastic line:
$$y'' = -\frac{M_x}{EI} = -\frac{-M_1 + \Sigma R_i x_i + P y_1}{EI} \qquad (2)$$

where E is the modulus of elasticity; I is the moment of inertia of member; M_1 and R_1, R_2 are reactions due to prestressing force; P is the prestressing force; y_1 is the eccentricity of the prestressing force at location x.

This differential Eq. (2) varies with x and has unknowns M_1 and R_i. Solving this equation with the boundary conditions, the engineers find out the relationship between the reactions, prestressing force, and eccentricity of the prestressing force. Combined with other equilibrium equations, the engineers obtain unknowns M_1 and R_i. Finally, the engineers determine the prestressing moment which is caused by prestressing force and reactions, and secondary moment which is caused only by reactions.

APPLICATIONS AND INDEPENDENT VERIFICATION

The two-span beam

The two-span beam and tendon have profiles of two parabolas y_1 and y_2 in Figure 2.

where $a_1 = \frac{4e_1}{L_1^2}$; $b_1 = -\frac{4e_1}{L_1}$; $a_2 = \frac{4e_2}{L_2^2}$; $b_2 = -\frac{4e_1}{L_2^2}(2L_1 + L_2)$; $c_2 = \frac{4e_2 L_1}{L_2^2}(L_1 + L_2)$

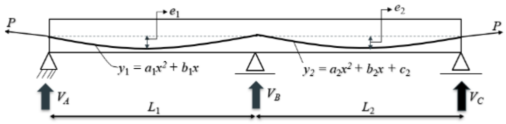


Figure 2. The two-span prestressed beam and reactions [Source: The Authors]

The equilibrium of the left segment (shear force is ignored) in Figure 3:

$$\Sigma F_X = P + N = 0 \Rightarrow N = -P \qquad (3)$$

$$\Sigma M = M_x + P|y_1| - V_A x = 0 \quad (4)$$
$$\Rightarrow M_x = P(a_1x^2 + b_1x) + V_A x$$

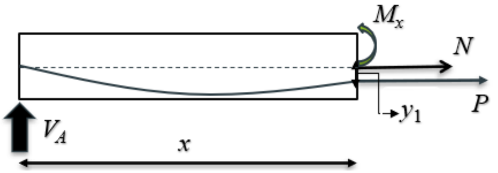


Figure 3. The equilibrium of the left segment from point A [Source: The Authors]

The differential equation of the elastic line and its antiderivative are as follows:

$$y'' = -\frac{M_x}{EI} = -\frac{1}{EI}(Pa_1x^2 + Pb_1x + V_A x) \quad (5)$$

$$y' = \int y'' = -\frac{1}{EI}\left(\frac{Pa_1x^3}{3} + \frac{Pb_1x^2}{2} + \frac{V_A x^2}{2}\right) + B \quad (6)$$

$$y = \int y' = -\frac{1}{EI}\left(\frac{Pa_1x^4}{12} + \frac{Pb_1x^3}{6} + \frac{V_A x^3}{6}\right) + Bx + C \quad (7)$$

Because the displacements at point A and B are equal to 0, this results in:

$$y_A = y_0 = 0 \Rightarrow C = 0 \quad (8)$$

$$y_B = y_{L_1} = -\frac{1}{EI}\left(\frac{Pa_1L_1^4}{12} + \frac{Pb_1L_1^3}{6} + \frac{V_A L_1^3}{6}\right) + BL_1 = 0 \quad (9)$$
$$\Rightarrow B = \frac{1}{EI}\left(\frac{Pa_1L_1^3}{12} + \frac{Pb_1L_1^2}{6} + \frac{V_A L_1^2}{6}\right)$$

The slope at point B (left) derives from Eq. (6):

$$\varphi_{B-left} = y'_{L1} = -\frac{1}{EI}\left(\frac{Pa_1L_1^3}{3} + \frac{Pb_1L_1^2}{2} + \frac{V_A L_1^2}{2}\right) + B \quad (10)$$

By substituting B from Eq. (9) into Eq. (10), this results in:

$$\varphi_{B-left} = -\frac{1}{EI}\left(\frac{Pa_1L_1^3}{4} + \frac{Pb_1L_1^2}{3} + \frac{V_A L_1^2}{3}\right) \quad (11)$$

Similarly, the equilibrium of the right segment (shear force is ignored) in Figure 4:

$$\Sigma F_X = P + N = 0 \Rightarrow N = -P \quad (12)$$

$$\Sigma M = -M_x - P|y_2| + V_C(L - x) = 0 \quad (13)$$
$$\Leftrightarrow M_x = Py_2 + V_C(L - x)$$
$$= P(a_2x^2 + b_2x + c_2) + V_C(L - x)$$

where $L = L_1 + L_2$

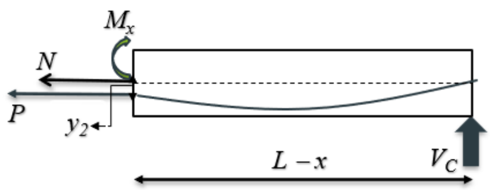


Figure 4. The equilibrium of the right segment from point A [Source: The Authors]

The differential equation of the elastic line and its antiderivative are as follows:

$$y'' = -\frac{M_x}{EI} = -\frac{1}{EI}[P(a_2x^2 + b_2x + c_2) + V_C(L - x)] \quad (14)$$

$$y' = \int y'' = -\frac{1}{EI}\left[\frac{Pa_2x^3}{3} + \frac{Pb_2x^2}{2} + Pc_2x - \frac{V_C(L-x)^2}{2}\right] + F \quad (15)$$

$$y = \int y' = -\frac{1}{EI}\left[\frac{Pa_2x^4}{12} + \frac{Pb_2x^3}{6} + \frac{Pc_2x^2}{2} + \frac{V_C(L-x)^3}{6}\right] + Fx + G \quad (16)$$

Because the displacements at point B and C are equal to 0, this results in:

$$y_B = -\frac{1}{EI}\left(\frac{Pa_2L_1^4}{12} + \frac{Pb_2L_1^3}{6} + \frac{Pc_2L_1^2}{2} + \frac{V_C L_2^3}{6}\right) + FL_1 + G = 0 \quad (17)$$

$$y_C = -\frac{1}{EI}\left(\frac{Pa_2L^4}{12} + \frac{Pb_2L^3}{6} + \frac{Pc_2L^2}{2}\right) + FL + G = 0 \quad (18)$$

From Eq. (17) and (18), F is figured out:

$$F = -\frac{1}{EI}\left[\frac{Pa_2(L_1^4 - L^4)}{12L_2} + \frac{Pb_2(L_1^3 - L^3)}{6L_2} + \frac{Pc_2(L_1^2 - L^2)}{2L_2} + \frac{V_C L_2^2}{6}\right] \quad (19)$$

The slope at point B (right) derives from Eq. (15):

$$\varphi_{B-right} = -\frac{1}{EI}\left(\frac{Pa_2L_1^3}{3} + \frac{Pb_2L_1^2}{2} + Pc_2L_1 - \frac{V_C L_2^2}{2}\right) + F \quad (20)$$

By substituting F from Eq. (19) into Eq. (20), this results in:

$$\varphi_{B-right} = -\frac{1}{EI}\left[\frac{Pa_2L_1^3}{3} + \frac{Pb_2L_1^2}{2} + Pc_2L_1 + \frac{Pa_2(L_1^4 - L^4)}{12L_2} + \frac{Pb_2(L_1^3 - L^3)}{6L_2} + \frac{Pc_2(L_1^2 - L^2)}{2L_2} - \frac{V_C L_2^2}{3}\right] \quad (21)$$

Because the slope at point B (or any point, cross section on continuous beam) must be the same, therefore:

$$\varphi_{B-left} = \varphi_{B-right}$$
$$\Leftrightarrow \frac{L_1^2}{3}V_A + \frac{L_2^2}{3}V_C = \frac{Pa_2L_1^3}{3} + \frac{Pb_2L_1^2}{2} + Pc_2L_1 + \frac{Pa_2(L_1^4 - L^4)}{12L_2} + \frac{Pb_2(L_1^3 - L^3)}{6L_2} + \frac{Pc_2(L_1^2 - L^2)}{2L_2} - \frac{Pa_1L_1^3}{4} - \frac{Pb_1L_1^2}{3} \quad (22)$$

Besides, the sum of moment of external forces about point B is:

$$\sum M_{/B} = M_{V_A/B} + M_{V_C/B} = -V_A L_1 + V_C L_2 = 0$$
$$\Rightarrow V_A = \frac{V_C L_2}{L_1} \quad (23)$$

By substituting V_A from Eq. (23) into Eq. (22), V_C is figured out:

$$V_C = \frac{Pa_2L_1^3}{L_2L} + \frac{3Pb_2L_1^2}{2L_2L} + \frac{3Pc_2L_1}{L_2L} + \frac{Pa_2(L_1^4 - L^4)}{4L_2^2L} + \frac{Pb_2(L_1^3 - L^3)}{2L_2^2L} + \frac{3Pc_2(L_1^2 - L^2)}{2L_2^2L} - \frac{3Pa_1L_1^3}{4L_2L} - \frac{Pb_1L_1^2}{L_2L} \quad (24)$$

Let $L_1 = 8$ m; $L_2 = 10$ m; $e_1 = 0.12$ m; $e_2 = 0.16$ m.

Tendons $a_1 = 0.0075$; $b_1 = -0.060$; $a_2 = 0.0064$; $b_2 = -0.1664$; $c_2 = 0.9216$; $P = 146$ N.

It is figured out that $V_C = 2.08$ N; $V_A = 2.59$ N.

Moment equations and their diagram (prestressing moment) in Figure 5(a). Diagram (secondary moment) in Figure 5(b):

segment 1 ($0 \le x \le 8$ m)

$$M_x = P\left(\frac{4e_1}{L_1^2}x^2 - \frac{4e_1}{L_1}x\right) + V_A x \quad (25)$$
$$= 1.09x^2 - 6.16x$$

segment 2 ($8 \le x \le 18$ m)

$$M_x = Py_2 + V_C(L_1 + L_2 - x) \quad (26)$$
$$= 0.93x^2 - 26.37x + 171.92$$

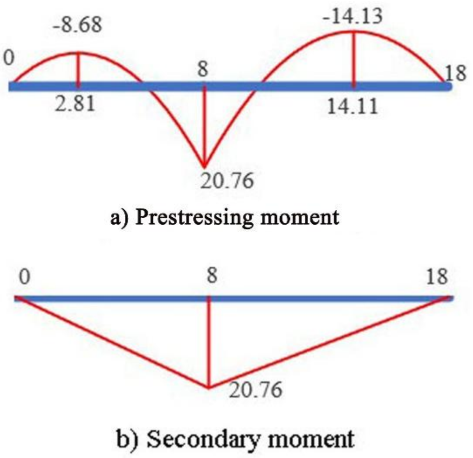


Figure 5. The moment diagrams of two-span beam (Nm) [Source: The Authors]

Single-span frame

The single-span frame and tendon have profiles of two segment parabolas y_1 and y_2 in Figure 6.

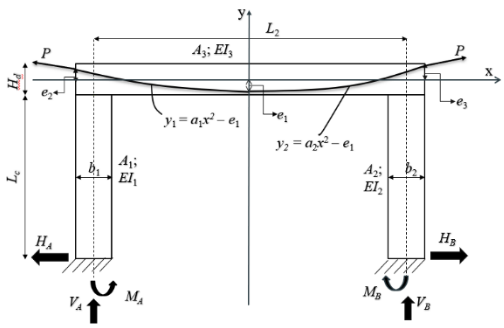


Figure 6. The single-span prestressed frame and reactions [Source: The Authors]

where $a_1 = \frac{4(e_1+e_2)}{(L_2+b_1)^2}$; $a_2 = \frac{4(e_1+e_3)}{(L_2+b_2)^2}$; $L_1 = L_c + \frac{H_d}{2}$; A_i is the cross-sectional area of beam or column; b_d and H_d are the width and height of beam respectively; h_1 and b_1 are the width and height of column A respectively; h_2 and b_2 are the width and height of column B respectively.

The equilibrium of external forces is presented as the following equations:

$$\Sigma F_X = -H_A + H_B = 0 \quad (27)$$

$$\Sigma F_Y = V_A + V_B = 0 \quad (28)$$

$$\Sigma M_{/A} = M_A + V_B L_2 - M_B = 0 \quad (29)$$

The engineers have six unknowns but only three equations; therefore, the engineers must find three more equations with the boundary conditions. After applying the prestressing force, the frame is deformed in Figure 7.

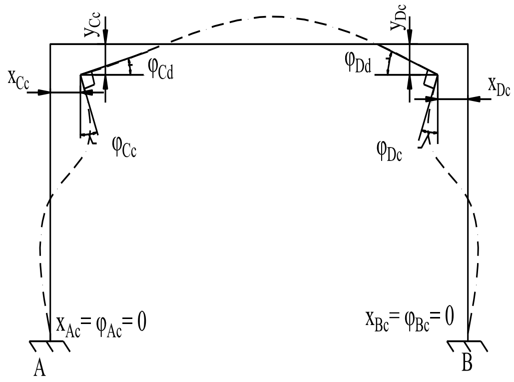


Figure 7. The deformations of the single-span frame after prestressing [Source: The Authors]

The fixed support A and B, boundary conditions are:

$$\varphi_{Ac} = 0 \quad (30)$$

$$y_{Ac} = 0 \quad (31)$$

$$\varphi_{Bc} = 0 \quad (32)$$

$$y_{Bc} = 0 \quad (33)$$

At point C and D. Before deformation, the angle between column and beam is 90 degrees. After deformation, these angle are also 90 degrees. Therefore, slope of column and beam are the same:

$$\varphi_{Cc} = \varphi_{Cd} \quad (34)$$

$$\varphi_{Dc} = \varphi_{Dd} \quad (35)$$

Axial deformation of column (due to axial force) induces vertical displacement at point C and D on beam, boundary conditions are:

$$\Delta L_{CO-A} = y_{Cc} \quad (36)$$

$$\Delta L_{CO-B} = y_{Dc} \quad (37)$$

Axial deformation of beam (due to axial force) pulls horizontal displacement at point C and D on colomn, boundary conditions are:

$$\Delta L_{Beam} = x_{Cc} + |x_{Dc}| \quad (38)$$

The equilibrium of column A in Figure 8(a):

$$\Sigma F_Y = N + V_A = 0 \Rightarrow N = -V_A \quad (39)$$

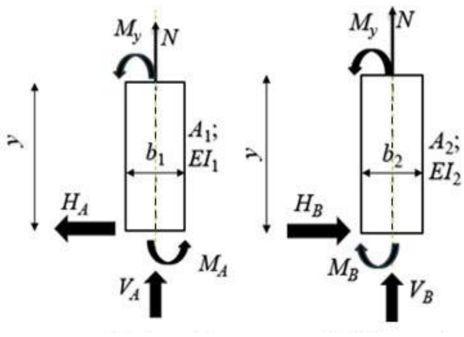
The axial deformation of column A

$$\Delta L_{CO-A} = \frac{N}{EA_1} L_1 = -\frac{V_A}{EA_1} L_1 \quad (40)$$

$$\Sigma M = M_A - H_A y + M_y = 0$$

$$\Rightarrow M_y = -M_A + H_A y$$

(41)



(a) Column A (b) Column B

Figure 8. The equilibrium of column [Source: The Authors]

The differential equation of the elastic line and its antiderivative are as follows:

$$x"=-\frac{M_y}{EI_1}=-\frac{1}{EI_1}(-M_A+H_Ay) \quad (42)$$

$$x'=\int x"=-\frac{1}{EI_1}(-M_Ay+\frac{H_Ay^2}{2})+B \quad (43)$$

$$x=\int x'=-\frac{1}{EI_1}(-\frac{M_Ay^2}{2}+\frac{H_Ay^3}{6})+By+G \quad (44)$$

Apply the (30) condition $\varphi_{Ac}=x'_0=0\Rightarrow B=0$

Apply the (31) condition $x_{Ac}=x_0=0\Rightarrow G=0$

It is figured out that:

$$\varphi_{Cc}=x'_{L_1}=-\frac{1}{EI_1}(-M_AL_1+\frac{H_AL_1^2}{2}) \quad (45)$$

$$x_{Cc}=x_{L_1}=\frac{1}{EI_1}(\frac{M_AL_1^2}{2}-\frac{H_AL_1^3}{6}) \quad (46)$$

Similarly, the equilibrium of column B in Figure 8(b):

$$\Sigma F_Y=N+V_B=0\Rightarrow N=-V_B \quad (47)$$

The axial deformation of column B

$$\Delta L_{CO-B}=\frac{N}{EA_2}L_1=-\frac{V_B}{EA_2}L_1 \quad (48)$$

$$\Sigma M=-M_B+H_By+M_y=0$$
$$\Rightarrow M_y=M_B-H_By \quad (49)$$

The differential equation of the elastic line and its antiderivative are as follows:

$$x"=-\frac{M_y}{EI_2}=-\frac{1}{EI_2}(M_B-H_By) \quad (50)$$

$$x'=\int x"=-\frac{1}{EI_2}(M_By-\frac{H_By^2}{2})+K \quad (51)$$

$$x=\int x'=-\frac{1}{EI_2}(\frac{M_By^2}{2}-\frac{H_By^3}{6})+Ky+F \quad (52)$$

Apply the (32) condition $\varphi_{Bc}=x'_0=0\Rightarrow K=0$

Apply the (33) condition $x_{Bc}=x_0=0\Rightarrow F=0$

It is figured out that:

$$\varphi_{Dc}=x'_{L_1}=-\frac{1}{EI_2}(M_BL_1-\frac{H_BL_1^2}{2}) \quad (53)$$

$$x_{Dc}=x_{L_1}=-\frac{1}{EI_2}(\frac{M_BL_1^2}{2}-\frac{H_BL_1^3}{6}) \quad (54)$$

The equilibrium of beam in Figure 9:

$$\Sigma F_X=N-H_A+P=0\Rightarrow N=H_A-P \quad (55)$$

$$\Sigma M=M_A-H_AL_1-V_A(x+\frac{L_2}{2})+M_x+P|y|=0 \quad (56)$$
$$\Rightarrow M_x=-M_A+H_AL_1+V_A(x+\frac{L_2}{2})+P(ax^2-e_1)$$

where $a=\begin{cases} a_1 \ if \ x\le 0 \\ a_2 \ if \ x\ge 0\end{cases}$

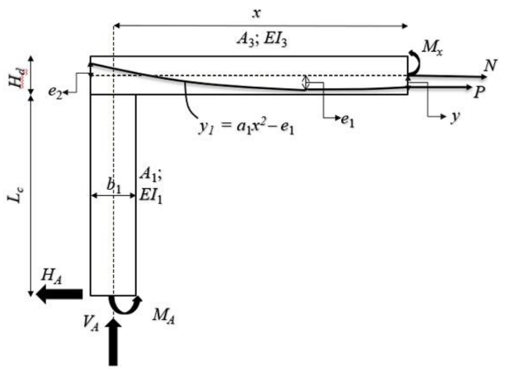


Figure 9. The equilibrium of column-beam A [Source: The Authors]

The differential equation of the elastic line is as follows:

$$y"=-\frac{M_x}{EI_3}=-\frac{1}{EI_3}[P(ax^2-e_1)+V_A(x+\frac{L_2}{2})+H_AL_1-M_A] \quad (57)$$

The slope equation:

$$y'=\int y"=-\frac{1}{EI_3}(\frac{Pax^3}{3}+\frac{V_Ax^2}{2}+\frac{V_AL_2}{2}x+H_AL_1x-M_Ax-Pe_1x)+M \quad (58)$$

The slope at the ends of beam:

$$y'_{-L_2/2}=-\frac{1}{EI_3}(-\frac{Pa_1L_2^3}{24}-\frac{V_AL_2^2}{8}-\frac{H_AL_1L_2}{2}+\frac{M_AL_2}{2}+\frac{Pe_1L_2}{2})+M \quad (59)$$

$$y'_{L_2/2}=-\frac{1}{EI_3}(\frac{Pa_2L_2^3}{24}+\frac{3V_AL_2^2}{8}+\frac{H_AL_1L_2}{2}-\frac{M_AL_2}{2}-\frac{Pe_1L_2}{2})+M \quad (60)$$

$$-\frac{1}{EI_3}(-\frac{Pa_1L_2^3}{24}-\frac{V_AL_2^2}{8}-\frac{H_AL_1L_2}{2}+\frac{M_AL_2}{2}+\frac{Pe_1L_2}{2})+M=-\frac{1}{EI_1}(-M_AL_1+\frac{H_AL_1^2}{2}) \quad (61)$$

Apply the (35) condition:

$$-\frac{1}{EI_3}(\frac{Pa_2L_2^3}{24}+\frac{3V_AL_2^2}{8}+\frac{H_AL_1L_2}{2}-\frac{M_AL_2}{2}-\frac{Pe_1L_2}{2})+M=-\frac{1}{EI_2}(M_BL_1-\frac{H_BL_1^2}{2}) \quad (62)$$

From Eq. (61) and (62), it is figured out that:

$$M=\frac{M_AL_1}{2EI_1}-\frac{M_BL_1}{2EI_2}-\frac{H_AL_1^2}{4EI_1}+\frac{H_BL_1^2}{4EI_2}+\frac{Pa_2L_2^3}{48EI_3}-\frac{Pa_1L_2^3}{48EI_3}+\frac{V_AL_2^2}{8EI_3} \quad (63)$$

And

$$-(\frac{L_2}{I_3}+\frac{L_1}{I_1})M_A-\frac{L_1}{I_2}M_B+\frac{L_2^2}{2I_3}V_A+(\frac{L_1L_2}{I_3}+\frac{L_1^2}{2I_1})H_A+\frac{H_BL_1^2}{2I_2}=-\frac{Pa_1L_2^3}{24I_3}-\frac{Pa_2L_2^3}{24I_3}+\frac{Pe_1L_2}{I_3} \quad (64)$$

The equation of the elastic line is as follows:

$$y=\int y'=-\frac{1}{EI_3}\left(\frac{Pax^4}{12}+\frac{V_Ax^3}{6}+\frac{V_AL_2x^2}{4}+\frac{H_AL_1x^2}{2}-\frac{M_Ax^2}{2}-\frac{Pe_1x^2}{2}\right)+Mx+N \quad (65)$$

The displacements at the ends of beam:

$$y_{-L_2/2}=-\frac{1}{EI_3}\left(\frac{Pa_1L_2^4}{192}+\frac{V_AL_2^3}{24}+\frac{H_AL_1L_2^2}{8}-\frac{M_AL_2^2}{8}-\frac{Pe_1L_2^2}{8}\right)-\frac{ML_2}{2}+N \quad (66)$$

$$y_{L_2/2}=-\frac{1}{EI_3}\left(\frac{Pa_2L_2^4}{192}+\frac{V_AL_2^3}{12}+\frac{H_AL_1L_2^2}{8}-\frac{M_AL_2^2}{8}-\frac{Pe_1L_2^2}{8}\right)+\frac{ML_2}{2}+N \quad (67)$$

Apply the (36) condition:

$$y_{-L_2/2}=y_{Cc}\Rightarrow-\frac{1}{EI_3}\left(\frac{Pa_1L_2^4}{192}+\frac{V_AL_2^3}{24}+\frac{H_AL_1L_2^2}{8}-\frac{M_AL_2^2}{8}-\frac{Pe_1L_2^2}{8}\right)-\frac{ML_2}{2}+N=-\frac{V_A}{EA_1}L_1 \quad (68)$$

Apply the (37) condition:

$$y_{L_2/2}=y_{Dc}\Rightarrow-\frac{1}{EI_3}\left(\frac{Pa_2L_2^4}{192}+\frac{V_AL_2^3}{12}+\frac{H_AL_1L_2^2}{8}-\frac{M_AL_2^2}{8}-\frac{Pe_1L_2^2}{8}\right)+\frac{ML_2}{2}+N=-\frac{V_B}{EA_2}L_1 \quad (69)$$

Subtract Eq. (68) from Eq. (69) then substitute M from Eq. (63):

$$-\frac{1}{EI_3}\left(\frac{Pa_2L_2^4}{192}-\frac{Pa_1L_2^4}{192}+\frac{V_AL_2^3}{24}\right)+ML_2=-\frac{V_B}{EA_2}L_1+\frac{V_A}{EA_1}L_1$$

$$\Leftrightarrow\frac{L_1L_2}{2I_1}M_A-\frac{L_1L_2}{2I_2}M_B-\frac{L_1^2L_2}{4I_1}H_A+\frac{L_1^2L_2}{4I_2}H_B+\left(\frac{L_2^3}{12I_3}-\frac{L_1}{A_1}\right)V_A+\frac{L_1}{A_2}V_B=-\frac{Pa_2L_2^4}{64I_3}+\frac{Pa_1L_2^4}{64I_3} \quad (70)$$

Apply the (38) condition and substitute N from Eq. (55):

$$x_{Cc}+|x_{Dc}|=\frac{NL_2}{EA_3}$$

$$\Leftrightarrow\frac{1}{I_1}\left(\frac{M_AL_1^2}{2}-\frac{H_AL_1^3}{6}\right)+\frac{1}{I_2}\left(\frac{M_BL_1^2}{2}-\frac{H_AL_1^3}{6}\right)=\frac{(P-H_A)L_2}{A_3}$$

$$\Leftrightarrow\frac{L_1^2}{2I_1}M_A+\frac{L_1^2}{2I_2}M_B+\left(\frac{L_2}{A_3}-\frac{L_1^3}{6I_1}-\frac{L_1^3}{6I_2}\right)H_A=\frac{PL_2}{A_3} \quad (71)$$

Solving the system of Eq.s (27), (28), (29), (64), (70), (71) the engineers obtain six unknowns.

Let:

$L_c = 2.70$ m; $L_1 = 3.00$ m; $L_2 = 7.60$ m.

Column A: $b_1 = 0.40$ m; $h_1 = 0.40$ m; $I_1 = 0.00213$ m^4; $A_1 = 0.16$ m^2.

Column B: $b_2 = 0.40$ m; $h_2 = 0.60$ m; $I_2 = 0.00320$ m^4; $A_2 = 0.24$ m^2.

Beam: $b_d = 0.30$ m; $h_d = 0.60$ m; $I_3 = 0.00540$ m^4; $A_3 = 0.18$ m^2.

Tendon: $e_1 = 0.20$ m; $e_2 = 0.10$ m; $e_3 = 0.20$ m; $a_1 = 0.0188$; $a_2 = 0.0250$; P = 146 N.

The engineers get a system of equations:

$$\begin{cases} -H_A+H_B=0 \\ V_A+V_B=0 \\ M_A-M_B+7.60V_B=0 \\ -2813.65M_A-937.50M_B+6331.60H_A+1406.25H_B+2674.07V_A=19460.81 \\ 5343.75M_A-3562.50M_B-8015.63H_A+5343.75H_B+33852.85V_A+12.50V_B=-8808.73 \\ 2109.38M_A+1406.25M_B-3473.40H_A=6164.44 \end{cases} \quad (72)$$

$$\Rightarrow\begin{cases} M_A = 7.49\text{ Nm} \\ M_B = 8.69\text{ Nm} \\ H_A = 6.29\text{ N} \\ H_B = 6.29\text{ N} \\ V_A = -0.16\text{ N} \\ V_B = 0.16\text{ N} \end{cases} \quad (73)$$

After solving the system of equations, the engineers can determine the prestressing and secondary moment diagrams in Figure 10 and Figure 11:

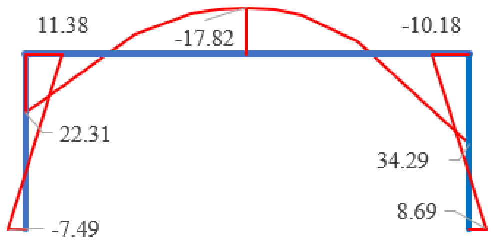


Figure 10. The prestressing moment diagram of frame (Nm)[Source: The Authors]

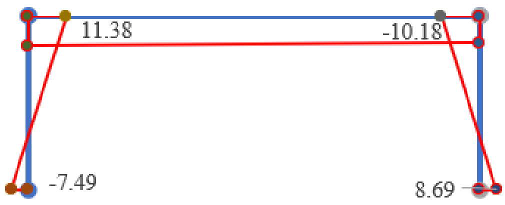


Figure 11. The secondary moment diagram of frame (Nm) [Source: The Authors]

Comparison with load balancing method

a) Methodology

Eq. (1) describes the equilibrium of the prestressed member, the engineers take the first and second derivatives of M_x with respect to x:

$$\frac{dM_x}{dx}=\Sigma R_i+P\frac{dy_1}{dx} \quad (74)$$

$$\frac{d^2M_x}{dx^2}=P\frac{d^2y_1}{dx^2}=w_b \quad (75)$$

If y_1 is a parabola described by the equation: $y_1 = ax^2 + bx + c$, Eq. (75) becomes:

$$\frac{d^2M_x}{dx^2}=2Pa=w_b \quad (76)$$

The load w_b in Eq. (76) is the equivalent load which T. Y. Lin has applied for analyzing prestressed concrete structures. For each type of tendon profile including straight, linear, cubic or any profile, the engineers can obtain the corresponding equivalent load. Especially, in the straight and linear profiles, the equivalent load obtained is equal to zero but Eq. (2) still describes the effect of prestressing force and reactions on curvature of members.

Mathematical theory shows that direct method is the opposite of load balancing method. The engineers derive from Eq. (1), direct method develops the differential Eq. (2) from Eq. (1) based on elastic theory, and then, by taking the antiderivatives, the equation is solved to obtain the reactions. On the other hand, load balancing method takes the second derivative of Eq. (1) to obtain the equivalent load w_b then applies this force to members. Finally, the results are convergent, and this theory affirms the accuracy of direct method.

Based on this theory, if tendon profile is $y_2 = ax^3 + bx^2 + cx + d$, the engineer finds out equivalent load describes $w_b = 3ax+b$ or any tendon profiles find out their equivalent load.

b) Verification and results comparison in terms of applications

In two-span beam and single-span frame, using load balancing method for analyzing, the procedure includes: applying balanced loading to beam, determining the reactions due to balanced loading, the prestressing moment due to balanced loading, the reactions due to prestressing force and finally the secondary moment due to reactions in Figure 12 and Figure 13.

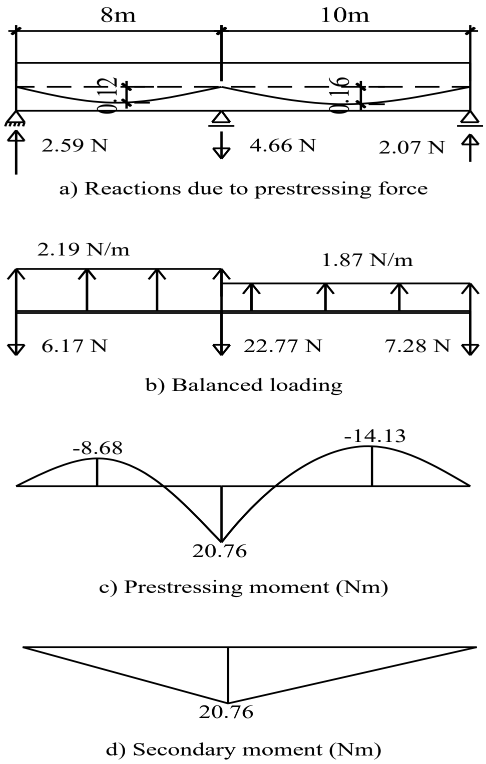


Figure 12. Analyzing two-span beam using load balancing method [Source: The Authors]

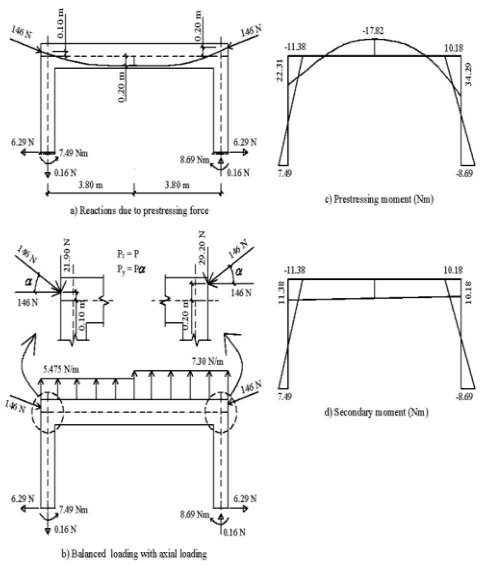


Figure 13. Analyzing single-span frame using load balancing method[13]

Comparing the results in table 1, direct method and load balancing method saw no differences in one-span beam and single-span frame. These results affirm the accuracy of direct method.

Table 1. Results comparison of direct method and load balancing method [Source: The Authors]

App s	Items	Direct metho d	Load balanc g method	Comparison s
Two-span beam	Prestressin g Reactions (VA; VB; VC order – Unit N)	2.59; 4.66; 2.07	2.59; 4.66; 2.07	0.00%; 0.00%; 0.00%
	Prestressin g moment	Fig. 5 (a)	Fig. 12 (c)	0.00%
	Secondary moment	Fig. 5 (b)	Fig. 12 (d)	0.00%
Single-span frame	Support A reactions (Vertical; Horizontal; Moment order – Unit N; N; Nm)	-0.16; 6.29; 7.49	-0.16; 6.29; 7.49	0.00%; 0.00%; 0.00%
	Support B reactions (Vertical; Horizontal; Moment order – Unit N; N; Nm)	0.16; 6.29; 8.69	0.16; 6.29; 8.69	0.00%; 0.00%; 0.00%
	Prestressin g moment	Fig. 10	Fig. 13 (c)	0.00%
	Secondary moment	Fig. 11	Fig. 13 (d)	0.00%

LARGE TENDON PROFILE SAG

In transfer or deep beams[14,15], small sag tendons are not appropriate. The prestressing force P in Figure 14 devides two components:

$$P_x = P\cos\alpha \quad (77)$$

$$P_y = P\sin\alpha \quad (78$$

Where

$$tg\alpha = \frac{dy_1}{dx} = y_1^{'} \quad (79)$$

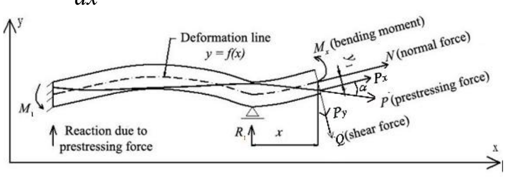


Figure 14. The equilibrium of prestressed member with large tendon profile sag [Source: The Authors]

Using the relationships:

$$\cos\alpha = \pm\sqrt{\frac{1}{1+tg^2\alpha}} = \pm\sqrt{\frac{1}{1+(y_1')^2}} \quad (80)$$

$$\sin\alpha = \pm\sqrt{\frac{tg^2\alpha}{1+tg^2\alpha}} = \pm\sqrt{\frac{(y_1')^2}{1+(y_1')^2}} \quad (81)$$

Eq. (2) becomes:

$$y'' = -\frac{M_x}{EI} = -\frac{-M_1 + \Sigma R_i x_i + \frac{P y_1}{\sqrt{1+(y_1')^2}}}{EI} \quad (82)$$

Shear force Q derive from:

$$Q = \sum R_i + P\sqrt{\frac{(y_1')^2}{1+(y_1')^2}};\ \text{if } y_1' \ge 0 \quad (83)$$

Or

$$Q = \sum R_i - P\sqrt{\frac{(y_1')^2}{1+(y_1')^2}};\ \text{if } y_1' < 0 \quad (84)$$

Application to the single-span transfer frame and tendon have profiles of parabola y = ax^2 - e in Figure 15(a).

The differential equation of the elastic line as follows:

$$y'' = -\frac{M_x}{EI_3} = -\frac{1}{EI_3}\left[\frac{Pax^2 - Pe}{\sqrt{1+4a^2x^2}} + V_A\left(x+\frac{L_2}{2}\right) + H_A L_1 - M_A\right] \quad (85)$$

The slope equation:

$$y' = \int y'' = -\frac{1}{EI_3}\left[\frac{Px}{4}\sqrt{x^2+\frac{1}{4a^2}} - \frac{P(8ae+1)}{16a^2}\ln\left|\sqrt{4a^2x^2+1}+2ax\right| + \frac{V_A}{2}x^2 + \frac{V_A L_2}{2}x + H_A L_1 x - M_A x)\right] + M \quad (86)$$

Equation of the elastic line:

$$y = \int y' = -\frac{1}{EI_3}\left[\frac{P}{12}\left(x^2+\frac{1}{4a^2}\right)^{\frac{3}{2}} - \frac{P(8ae+1)}{16a^2}\left(x\ln\left|\sqrt{4a^2x^2+1}+2ax\right| - \sqrt{x^2+\frac{1}{4a^2}}\right) + \frac{V_A}{6}x^3 + \frac{V_A L_2}{4}x^2 + \frac{H_A L_1 x^2}{2} - \frac{M_A x^2}{2}\right] + Mx + N \quad (87)$$

Let:

L_c = 7.00 m; L_1 = 8.25 m; L_2 = 7.60 m.

Column A: b_1 = 1.00 m; h_1 = 1.00 m; I_1 = 0.0833 m^4; A_1 = 1.00 m^2.

Column B: b_2 = 1.00 m; h_2 = 2.00 m; I_2 = 0.1667 m^4; A_2 = 2.00 m^2.

Transfer beam: b_d = 1.00 m; h_d = 2.50 m; I_3 = 1.302 m^4; A_3 = 2.50 m^2.

Tendon: e = 0.80 m; a = 0.0865; P = 146 N.

Solving the Eq.s (86) and (87) with boundary conditions, combined with other equilibrium equations, the engineers obtain unknowns M_A, M_B, V_A, V_B, H_A and H_B. Finally, determine the prestressing, secondary moment in Figure 15(b) and Figure 15(c).

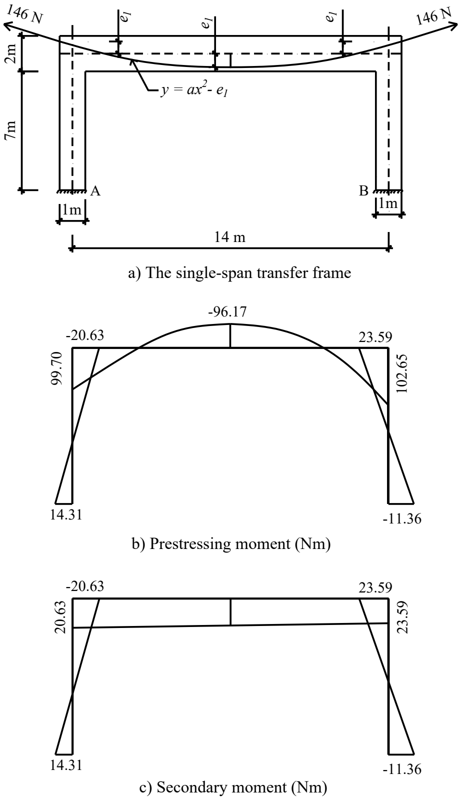


Figure 15. The single-span transfer frame with large tendon sag [Source: The Authors]

A wide range of sags are investigated in order to assess effect of large tendon profile compared with small tendon profile in Figure 16. The differences obtain clearly when ratio 2e/L_2 is increased. Analyzing large tendon sag reduces horizontal component and increases vertical component of prestressing force, this is made for reducing secondary moment (reactions) and prestressing moment; increasing shear force. The light reduction of secondary moment (less than 6%) is showed in Figure 16 .

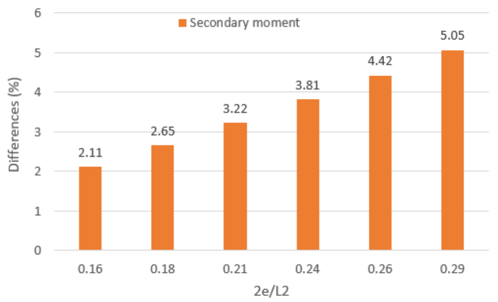


Figure 16. Maximum differences between small and large tendon sag [Source: The Authors]

CONCLUSIONS

Based on this method and its results, the following conclusions are drawn:

- The direct method provides more general way for analyzing prestressed concreate structures. The equivalent load models associated with any complicated tendon profiles which are formularized deal with and expand the unconventional tendon profiles. The consideration of frictional loss carries out based on this method, when the prestressing forces along the tendon(located x) are the function $P_x = P_0 e^{-(\mu\alpha + kx)}$ thus the equilibrium is taken P_x directly not replaced force (approximate force) in conventional method.

- This method accept that the eccentricity of the prestressing force is both small and large in comparison to the span. It finds out solution for the large tendon sag in transfer or deep beam.

- The single-span transfer frame with large tendon sag is investigated, the differences between small and large tendon sag depend on tendon sag, the stiffness of column and transfer beam. However, the differences are small. When ratio eccentricity e and clear span L (2e/L>0.2) is the difference of secondary moment more than 5%.

CONFLICT OF INTEREST

The authors declare that: in this research, there is no conflict of interests.

AUTHORS' CONTRIBUTION

Two authors are similar contribution in this research.

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Bài Nghiên Cứu

PHƯƠNG PHÁP TRỰC TIẾP TRONG PHÂN TÍCH KẾT CẤU BÊ TÔNG ỨNG SUẤT TRƯỚC

Phạm Thành Công[1,*], Vũ Việt Hoàng[2]

TÓM TẮT

Một trong những tiến bộ quan trọng của ngành xây dựng là bê tông dự ứng lực. Phạm vi ứng dụng rất rộng rãi trong các tòa nhà cao tầng, các công trình có nhịp lớn và cầu. Bê tông ứng suất trước đã được sử dụng thành công cho cả các dự án nhỏ và lớn hơn sáu mươi năm qua. Hiệu quả của nó đến từ khả năng sử dụng vật liệu cường độ cao, tận dụng tối đa toàn bộ mặt cắt ngang, điều chỉnh lực và vị trí cốt thép để chịu tải trọng tốt nhất, và kiểm soát thời điểm tác dụng lực ứng suất trước lên kết cấu. Trong kết cấu bê tông dự ứng lực, tải trọng cân bằng là một phương pháp gián tiếp và quen thuộc để phân tích các cấu kiện dự ứng lực thông qua tải trọng tương đương. Khung chịu tải trọng tương đương sinh momen cân bằng hoặc momen ứng suất trước (được tính đến trong trạng thái giới hạn khả năng sử dụng) và các phản lực; sau đó, momen thứ cấp (được tính đến trong trạng thái giới hạn chịu lực) được sinh ra từ các phản lực. Bài báo này trình bày một phương pháp trực tiếp để phân tích các cấu kiện dự ứng lực bằng cách giải các phương trình vi phân của đường đàn hồi. Các phương trình vi phân này xem xét sự cân bằng của các phản lực và độ lệch tâm của lực ứng suất trước. Sau khi giải các phương trình này, các kỹ sư có thể trực tiếp thu được các phản lực; và sau đó, xác định được momen ứng suất trước, momen thứ cấp và lực cắt. Phương pháp trực tiếp này cũng tìm ra ảnh hưởng chính xác của cáp có độ trùng lớn trong dầm chuyển hoặc dầm sâu. Ảnh hưởng cáp có độ trùng lớn tương tự như cáp độ trùng bé, sự khác biệt rất nhỏ. Ngoài ra, trong bài báo này, các mô hình tải trọng tương đương liên quan đến dạng bất kỳ của quĩ đạo cáp phức tạp được công thức hóa để giải quyết và mở rộng khả năng ứng dụng của các hình dạng dây cáp phi truyền thống trong tương lai của ngành xây dựng.

Từ khoá: Phương trình vi phân của đường đàn hồi, cáp độ trùng lớn, momen ứng suất trước, momen thứ cấp, phản lực do lực ứng suất trước, kết cấu bê tông ứng suất trước, phân tích kết cấu

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