VNUHCM Journal of Engineering and Technology 2026, 9(3): 3312-3325

Open Access Full Text Article

Research article

SOME EXISTENCE RESULTS FOR KIRCHHOFF DOUBLE PHASE PROBLEMS INVOLVING SUPERLINEAR AND CRITICAL GROWTHS

Hoang Hai Ha [1,2,*], Doan Thi Thanh Xuan [3]

Use your smartphone to scan this QR code and download this article

ABSTRACT

In this paper, we aim at investigating a class of nonlinear elliptic problems involving double-phase operators with variable exponents. More precisely, we consider differential operators of the form

$$div\ (|\xi|^{p(x)-2}\xi + \mu(x)|\xi|^{q(x)-2}\xi)$$

together with a nonlocal term. Such problems arise in a variety of applications, including reaction–diffusion processes, nonlinear elasticity, non-Newtonian fluid mechanics, and quantum physics.

One of the distinguishing features of the double-phase operator is its ability to continuously switch between different elliptic phases depending on the spatial position. This phenomenon significantly increases the analytical complexity of the associated problems and distinguishes them from classical (p)-Laplacian models. The reaction term considered in this work combines both subcritical and critical growth nonlinearities, thereby encompassing a broader class of equations than those treated in many previous studies.

Our main objective is to establish the existence of weak solutions and to analyze their qualitative properties. The problem is studied in an appropriate Musielak–Orlicz–Sobolev space associated with the double-phase structure. The variational approach serves as the primary analytical framework, allowing the solutions to be characterized as critical points of the corresponding energy functional. To this end, we verify the geometric conditions required by critical point theory and investigate the compactness properties of the functional. While compactness follows directly from Sobolev-type embeddings in the subcritical case, the presence of critical growth leads to a loss of compactness. To overcome this difficulty, we employ a concentration–compactness principle of Lions type adapted to the present setting, which enables us to recover the compactness necessary for the variational analysis. Our results contribute and complement several existing results and provide a flexbile variational framework that can be applied to classes with similar differential operators.

Key words: ``double phase operators'', ``Kirchhoff double phase problems'', ``variable exponents'', ``critical growths''

[1]*Department of Applied Mathematics, Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268 Ly Thuong Kiet Street, Dien Hong Ward, Ho Chi Minh City, Vietnam*

[2]*Vietnam National University Ho Chi Minh City, Linh Xuan Ward, Ho Chi Minh City, Vietnam*

[3]*Industrial University of Ho Chi Minh City, 12 Nguyen Van Bao Street, Hanh Thong Ward, Ho Chi Minh City, Vietnam*

Correspondence

Hoang Hai Ha, Department of Applied Mathematics, Faculty of Applied Science, Ho Chi Minh City University of Technology (HCMUT), 268 Ly Thuong Kiet Street, Dien Hong Ward, Ho Chi Minh City, Vietnam

Vietnam National University Ho Chi Minh City, Linh Xuan Ward, Ho Chi Minh City, Vietnam

Email: hoanghaiha@hcmut.edu.vn

History

- Received: 25-02-2026
- Revised: 08-04-2026
- Accepted: 05-06-2026
- Published Online: 11-09-2026

DOI : **https://doi.org/10.32508/vnuhcmj-et.v9i3.1577**

Check for updates

INTRODUCTION

This work is devoted to examining some existence results for the following Kirchhoff-type problems, involving double phase operator

$$\begin{cases} -K\left(\int_{\Omega}\mathcal{T}(x,\nabla u)dx\right)\mathrm{div}T(x,\nabla u) = \lambda f(x,u) + \theta g(x,u)\ in\ \Omega \\ u = 0\ on\ \partial\Omega \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^{\mathrm{d}}$ $(d \geq 2)$ denotes a bounded domain whose the boundary is Lipschitz; the functions $\mathcal{T}: \overline{\Omega} \times \mathbb{R}^{d} \to \mathbb{R}$ and $T: \overline{\Omega} \times \mathbb{R}^{d} \to \mathbb{R}^{d}$ are defined by

$$\mathcal{T}(x,\xi) := \frac{|\xi|^{p(x)}}{p(x)} + \mu(x)\frac{|\xi|^{q(x)}}{q(x)}$$

and

$$T(x,\xi) := \nabla_{\xi}\mathcal{T}(x,\xi) = |\xi|^{p(x)-2}\xi + \mu(x)|\xi|^{q(x)-2}\xi$$

where $p\ and\ q \in C_{+}(\overline{\Omega}) := \left\{ l \in C(\overline{\Omega}), 1 < l^{-} := \min_{\xi\in\overline{\Omega}} l(\xi) \leq l^{+} := \max_{\xi\in\overline{\Omega}} l(\xi) \right\}; \mu(\cdot) \geq 0;$

$K: [0,\infty) \to \mathbb{R}$, which is called Kirchhoff term; λ, θ are positive real parameters; $f(x,u)\ is$ a Carathéodory function involving superlinear growths; while $g(x,u)$ is the function given by

$$g(x,u) := b_1(x)|u|^{s_1(x)-2}t + b_2(x)\mu(x)^{\frac{s_2(x)}{q(x)}}|u|^{s_2(x)-2}u \text{ for } (x,u) \in \overline{\Omega} \times \mathbb{R}$$

Cite this article : Ha H H, Thanh Xuan D T. . **SOME EXISTENCE RESULTS FOR KIRCHHOFF DOUBLE PHASE PROBLEMS INVOLVING SUPERLINEAR AND CRITICAL GROWTHS.** *VNUHCM J. Eng. Technol. 9(3): 3312-3325.*

where $b_1(\cdot) > 0, b_2(\cdot) \geq 0$ such that $b_1, b_2 \in L^\infty(\Omega)$, exponents $s_1, s_2 \in C_+(\overline{\Omega})$ and possibly reach the critical thresholds.

Since the pioneering research of (Zhikov, 1987) on energy functionals with non-homogenous density in the context of elastic theory, problems involving operators so-called "double phase" have been an interesting topic for numerous researchers. Boundary value problems with double phase operators, beyond their profound applications in many fields (reaction-diffusion models, fluid mechanics, quantum physic…,) see (Bahrouni et al., 2019; Benci et al., 2000; Cherfils & Il'yasov, 2005) for more details, pose significant differences compared to problems with homogenous elliptic operators as existing p-Laplacian problems.

Primary studies following Zhikov's work focused on the minimization of energy functionals problems related to density functions satisfying unbalanced growth conditions. One notable functional among these is presented by

$$\int_\Omega [|\nabla u|^p + \mu(x)|\nabla u|^q]dx, \tag{2}$$

with $\Omega \subset \mathbb{R}^d$ being a bounded domain, the constants $p \,\&\, q \in (1, d)$ satisfying $p < q$ and $\mu(\cdot)$ is a nonnegative weight, see papers (Baroni et al., 2018; Byun & Oh, 2020; Marcellini, Paolo, 1989) and references therein. One important characteristic of such functionals (2) lies in the change continuously between two phases $p \,\&\, q$ according to the spatial point.

One can observe that problems on minimizing functionals (2) are identified with the studies of solutions to the associated Euler-Lagrange equation given by

$$-\mathrm{div} T(x, \nabla u) = 0 \quad in\ \Omega$$

when exponents p, q are constants. A natural research direction that follows is the investigation of

$$-\mathrm{div} T(x, \nabla u) = f(x, u)\ in\ \Omega \tag{3}$$

with various conditions of the nonlinear term $f(x, u)$. The paper (Colasuonno & Squassina, 2016) contributed a remarkable result on the core properties of spaces "Musielak-Orlicz-Sobolev" type, defined for "double phase" problems. Accordingly, further research on solutions of (3) have been developed rapidly, leading to numerous publications.

In general, approaches used to study problem (3) depend strongly on the conditions imposed on the reaction term. Due to the "compact embedding" result of $W_0^{1,\mathcal{M}_c}(\Omega)$ (see Section 3 for the definition) into $L^\alpha(\Omega)$, where $\mathcal{M}_c(x,t) := t^p + \mu(x)t^q$ for $(x,t) \in \Omega \times (0, \infty)$ and $1 \leq \alpha < p^* := \frac{dp}{d-p}$, several papers addressed the existence of solutions to problem (3) with nonlinearities of growths below p^* (subcritical growths). Even in the case of subcritical growth, approaches are diverse. We refer to (Liu & Dai, 2018a, 2018b) (Nehari manifold, Fountain and Dual Fountain theorems), (Perera & Squassina, 2018) (Morse theory), (Farkas et al., 2021) (Finsler manifold) and various references therein. Building upon studies with constant exponents, recent contributions have been extended to variable exponent cases, and moreover, with the presence of nonlinearities involving critical growths. The framework of spaces "Musielak-Orlicz-Sobolev" type related to operators $\mathrm{divT}(x, \nabla u)$ was introduced in (Crespo-Blanco et al., 2022). Following this, several results for double phase problems with variable exponents have been derived. Considering problem *(1)* with $K = 1$, we refer to (Ho & Winkert, 2023b), (Zeng et al., 2022) with some interesting results. The inclusion of a Kirchhoff-type term in double phase problems is motivated by Kirchhoff's original idea for dynamic problems describing vibration of an elastic string. Extending from the classical wave equation, Kirchhoff formulated models in which the tension depends on the energy of the entire string, leading to the "nonlocal" feature. Kirchhoff-type problems driven by the "double phase" operators with constant exponents have been studied intensively in works (Cen et al., 2023; Fiscella et al., 2024; Fiscella & Pinamonti, 2023) and references therein. Developments to those with variable exponents have been discussed after that. More precisely, in (Ho & Winkert, 2023a), the authors considered problem *(1)* with $\theta = 0$ and $p-$sublinear growth condition of $f(x,t)$. They provided the existence of multiple solutions with small energy, basing on a result about the boundedness for solutions to double phase problems in bounded domains, see (Ho & Winkert, 2023b). For further results on the existence of solutions to Kirchhoff problems with double phase operators involving variable exponents, we refer the reader to recent papers (Bouaam et al., 2025; El Ahmadi et al., 2024; Zuo & Massar, 2024), where conditions involving nonlinearities were considered variously (superlinear growth, logarithmic form,…). One common setting for nonlinearities in the aforementioned articles is that nonlinearities satisfy subcritical growths (below $p^*(\cdot)$), while problems with critical growth settings have been less studied.

Double phase problems of Kirchhoff-type with the presence of a nonlinearity satisfying critical growths were addressed in some recent works (Colasuonno & Perera, 2025; Ha & Ho, 2024). Particularly, the "concentration compactness principle" (abbreviated as CCP) of Lions-type was established in (Ha & Ho, 2024) to overcome the lack of compactness. A notable aspect in (Ha & Ho, 2024) is about CCP construction with the threshold of critical terms general than that in the foregoing papers. Those critical terms were introduced for the first time in (Ho & Winkert, 2023b), aiming to provide better description of critical Sobolev embeddings for spaces driven by double phase operators.

Motivated by the recent progress on Kirchhoff double phase problems, we aim at contributing some existence results for Kirchhoff critical double phase problems *(1)*, employing ideas used by Chung-Ho in (Chung & Ho, 2022) and Ho-Sim in (Ho & Sim, 2023). Our main results are twofold: the existence of a nontrivial solution for Kirchhoff problem *(1)* in **Theorem 1**, while the existence of multiple solutions for that with constant Kirchhoff term is stated in **Theorem 2**. In both cases, the conditions imposed on the subcritical term includes type of q −superlinear growth. To deal with the lack of compactness triggered by the presence of critical term, we exploit the mentioned CCP result in (Ha & Ho, 2024).

We outline the remainder as follows: our main results are presented in Section 2, Section 3 involves mathematical backgrounds, while detailed proofs of our main results are included in Section 4.

MAIN RESULTS

This section presents our two results on the finding of a nontrivial solution and multiple solutions. Before stating them, let us introduce briefly the definition and some crucial properties of spaces type of Musielak-Orlicz-Sobolev, associated with double phase operators. For this scope, we introduce the following function of double phase form:

$$\Psi(x,t) := w_1(x)t^{r_1(x)} + w_2(x)t^{r_2(x)}, \tag{4}$$

where $r_1, r_2 \in C_+(\overline{\Omega})$ with $r_1(\cdot) < r_2(\cdot)$, $0 < w_1 \in L^\infty(\Omega)$ and $0 \le w_2 \in L^\infty(\Omega)$. The Musielak-Orlicz spaces associated with Ψ are

$$L^\Psi(\Omega) := \{u \text{ is measurable on } \Omega \text{ such that } \rho_\Psi(u) < \infty\},$$

where $\rho_\Psi(u)$ is the modular associated with Ψ and defined by

$$\rho_\Psi(u) := \int_\Omega \Psi(x,|u|)dx. \tag{5}$$

We equip $L^\Psi(\Omega)$ with the Luxemburg norm given by

$$||u||_\Psi := inf\left\{\tau > 0 : \rho_\Psi\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Recalling from Propositions 2.6 and 2.12 of (Crespo-Blanco et al., 2022), the space $L^\Psi(\Omega)$ is a separable and reflexive Banach space. To look for solutions of problem *(1)*, we define the Musielak-Orlicz-Sobolev space $W^{1,\Psi}(\Omega)$ as

$$W^{1,\Psi}(\Omega) := \{\, u \in L^\Psi(\Omega) \text{ and } |\nabla u| \in L^\Psi(\Omega)\}$$

and we also equip $W^{1,\Psi}(\Omega)$ with the norm

$$||u||_{1,\Psi} := ||u||_\Psi + ||\,|\nabla u|\,||_\Psi.$$

We denote by $W_0^{1,\Psi}(\Omega)$ the completion of $C_c^\infty(\Omega)$ in $W^{1,\Psi}(\Omega)$. In view of Proposition 2.12 (Crespo-Blanco et al., 2022), $W^{1,\Psi}(\Omega)$ and $W_0^{1,\Psi}(\Omega)$ are reflexive Banach spaces.

Throughout the paper, we define the functions $\mathcal{M}$ and $\mathcal{G}$ as follows

$$\mathcal{M}(x,t) := t^{p(x)} + \mu(x)t^{q(x)} \tag{6}$$

and

$$\mathcal{G}(x,t) := b_1(x)t^{s_1(x)} + b_2(x)\mu(x)^{\frac{s_2(x)}{q(x)}} t^{s_2(x)} \tag{7}$$

for all $x \in \overline{\Omega}$ *and* $t \geq 0$.

We impose the following assumptions on data:

$(\mathcal{A}_1)$ μ, p, q are Lipschitz continuous functions satisfying $\mu(\cdot) \geq 0, 1 < p(\cdot) < q(\cdot) < d$ and $q(\cdot) < p(\cdot)\left(1 + \frac{1}{d}\right)$.

$(\mathcal{A}_2)$ $b_1, b_2 \in L^\infty(\Omega)$ and $s_1, s_2 \in C_+(\overline{\Omega})$ such that $b_1(x) > 0, b_2(x) \geq 0, q(x) < s_1(x) \leq p^*(x)$, $p^*(x) - s_1(x) = q^*(x) - s_2(x)$ for all $x \in \overline{\Omega}$ and

$$\Pi := \{ x \in \overline{\Omega}:\ s_1(x) = p^*(x),\ \ s_2(x) = q^*(x)\} \neq \emptyset.$$

The condition $(\mathcal{A}_1)$ guarantees the critical embedding for the solution space $W_0^{1,\mathcal{M}}(\Omega)$, which is crucial for our analysis due to the presence of critical exponents. Moreover, under $(\mathcal{A}_2)$, the concentration compactness principle holds, which is essential to handle the loss of compactness, (see Section 3 for details).

For our existence result, we require the following assumptions.

(K) The function $K: [0, +\infty) \to \mathbb{R}$ is continuous, non-decreasing on the interval $[0, t_0)$ for some $t_0 > 0$ and $K_0 := K(0) > 0$.

(P)
$$\kappa := inf_{v \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^{p(x)}\, dx}{\int_\Omega |v|^{p(x)}\, dx} > 0.$$

(f_0) There exist a function $m \in C_+(\overline{\Omega})$ such that $q^+ < m^- \leq m(x) < s_1(x)$ for all $x \in \overline{\Omega}$ and a positive constant $\overline{C}$ such that

$$|f(x,t)| \leq \overline{C}\left(1 + |t|^{m(x)-1}\right) \text{ for a.a. } x \in \overline{\Omega} \text{ and } t \in \mathbb{R}.$$

(f_1) There exists a constant $\sigma \in (q^+, s_1^-)$ such that $F(x,t) > 0$ and

$$tf(x,t) - \sigma F(x,t) \geq 0 \text{ for } a.a.\, x \in \overline{\Omega} \text{ and } t \in \mathbb{R},$$

where $F(x,t)$ denotes a primitive of $f(x,t)$ with respect to t.

(f_2) $\frac{f(x,t)}{|t|^{p(x)-1}} \to 0$ as $t \to 0$ uniformly for almost all $x \in \overline{\Omega}$.

Remark that if $p(\cdot) \equiv constant$ then (P) holds automatically. If $p(\cdot) \not\equiv constant$, then (P) satisfies if there exists $\xi \in \mathbb{R}^d \setminus \{0\}$ such that for any $x \in \Omega$, the function $\Theta(\tau) := \Theta(x + \tau\xi)$ with $\tau \in \{\tau \in \mathbb{R}, x + \tau\xi \in \Omega\}$ is monotone. Moreover, (P) does not fulfill if there is an open ball $B(x_0, \epsilon)$ centered at x_0 with radius ϵ such that $p(x_0) < p(x)$ (or $p(x_0) > p(x)$) for all $x \in \partial B(x_0, \epsilon)$, see Theorems 3.1 and 3.3 (Fan et al., 2005) for more details. Notice that from (P) one has

$$\kappa \int_\Omega |v|^{p(x)} \leq \int_\Omega |\nabla v|^{p(x)} \text{ for any } v \in W_0^{1,\mathcal{M}}(\Omega) \tag{8}$$

Solutions of problem *(1)* will be looked for in the functional space W:=$W_0^{1,\mathcal{M}}(\Omega)$, on which we use the norm $||\cdot||$ as $|||\nabla\cdot|||_{\mathcal{M}}$ (Proposition 2.21, (Crespo-Blanco et al., 2022)). We give the definition of a "weak solution" for *(1)* as follows. We say that a function $\tilde{u} \in W$ is a (weak) solution of problem *(1)* if

$$K\left(\int_\Omega \mathcal{T}(x, \nabla\tilde{u})dx\right)\int_\Omega T(x, \nabla\tilde{u}) \cdot \nabla\varphi dx - \lambda\int_\Omega f(x, \tilde{u})\varphi dx - \theta\int_\Omega g(x, \tilde{u})\varphi dx = 0$$

for any $\varphi \in W$.

We state our first result as follows.

Theorem 1 *Let $\theta = 1$ and $(\mathcal{A}_1) - (\mathcal{A}_2)$, (K), (P), $(f_0) - (f_2)$ be satisfied. Moreover, assume that $b_1 \in L^{-\frac{m(\cdot)}{s_1(\cdot)-m(\cdot)}}(\Omega)$. Then there exists a positive λ^* such that problem (1) possesses a nontrivial solution u_λ whenever $\lambda \in [\lambda^*, \infty)$. Furthermore, this solution u_λ satisfies*

$$\lim_{\lambda\to\infty} ||u_\lambda|| = 0. \tag{9}$$

In addition, we obtain a multiplicity result if we replace (f_2) by the condition of locally superlinear growths at infinity as follows.

(f_3) For some ball $\mathcal{B} \subset \overline{\Omega}$, there holds $lim_{t\to+\infty} \frac{f(x,t)}{t^{q_{\mathcal{B}}^+}} = +\infty$ uniformly $x \in \mathcal{B}$,

where $q_{\mathcal{B}}^+ := \max_{x\in\mathcal{B}} q(x)$.

Theorem 2 *Assume that $(\mathcal{A}_1) - (\mathcal{A}_2)$, (P), $(f_0) - (f_1)$ and (f_3) hold. Additionally, suppose that $f(x,t)$ is odd in t. Then for each given $\lambda > 0$ and each $j \in N$, there exists $\theta_j > 0$ such that problem (1) with $K = 1$ admits at least j pairs of nontrivial solutions whenever $\theta \in (0, \theta_j)$.*

One version of the last theorem for weighted $p(\cdot) -$ Laplacian problems can be found in Theorem 1.2 (Ho & Sim, 2023).

PRELIMINARIES

We consider $\Omega \subset \mathbb{R}^d$ as a bounded domain with Lipschitz boundary $\partial\Omega$ and the function Ψ given in (4). We already defined spaces $L^\Psi(\Omega)$, $W_0^{1,\Psi}(\Omega)$ with corresponding norms $||\cdot||_\Psi$, $||\cdot||_{1,\Psi}$. Note that, if we set $w_2 \equiv 0$, the space $L^\Psi(\Omega)$ returns to the well-known weighted Lebesgue space with variable exponent $L^{r_1(\cdot)}(w_1, \Omega)$ equipped with the norm $||\cdot||_{L^{r_1(\cdot)}(w_1,\Omega)}$, whose crucial properties can be referred to (Diening et al., 2011; Ho & Sim, 2016). Furthermore, if $w_1 \equiv 1$ we write $L^{r_1(\cdot)}(\Omega)$ and $||\cdot||_{r_1(\cdot)}$ in the place of $L^{r_1(\cdot)}(w_1, \Omega)$ and $||\cdot||_{L^{r_1(\cdot)}(w_1,\Omega)}$, respectively.

The next proposition involves the relation between modular and norm for $L^\Psi(\Omega)$, see (Crespo-Blanco et al., 2022; Ha & Ho, 2024).

Proposition 1

Assuming $u \in L^\Psi(\Omega)$, it holds that:

(i). if $u \neq 0$ *and* $||u||_\Psi = \lambda \Leftrightarrow \rho_\Psi\left(\frac{u}{\lambda}\right) = 1$

(ii). $\rho_\Psi(u) < 1$ $(>, = 1, \text{resp.}) \Leftrightarrow ||u||_\Psi < 1$ $(>, = 1, \text{resp.})$

(iii). $\min\left\{||u||_\Psi^{r_1^-}, ||u||_\Psi^{r_2^+}\right\} \le \rho_\Psi(u) \le \max\left\{||u||_\Psi^{r_1^-}, ||u||_\Psi^{r_2^+}\right\}$.

As a result, if $\{u_n\}_{n\in N} \subset L^\Psi(\Omega)$, then

$$\lim_{n\to\infty} \rho_\Psi(u_n) = 0 \Leftrightarrow \lim_{n\to\infty} ||u_n||_\Psi = 0.$$

We define $\mathcal{M}$ and $\mathcal{G}$ as in (6) and (7), respectively. In what follows, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ always are satisfied. Note that $(\mathcal{A}_2)$ implies

$$s_1(\cdot) \le s_2(\cdot) \le q^*(\cdot).$$

Recalling the Sobolev conjugate exponent of $t(\cdot) < d$:

$$t^*(\cdot) := \frac{dt(\cdot)}{d - t(\cdot)}.$$

In the sequel, notations $W_1 \hookrightarrow W_2$ and $W_1 \hookrightarrow\hookrightarrow W_2$ indicate respectively the continuous and compact embeddings from W_1 into W_2. The following results are found in (Crespo-Blanco et al., 2022).

Proposition 2 (Crespo-Blanco et al., 2022)

(i). If $\alpha(\cdot)$ is continuous and $1 \le \alpha(\cdot) \le p^*(\cdot)$ then $W^{1,\mathcal{M}}(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega)$.

(ii). If $\alpha(\cdot)$ is continuous and $1 \le \alpha(\cdot) < p^*(\cdot)$ then $W^{1,\mathcal{M}}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha(\cdot)}(\Omega)$.

Proposition 2.21 (Crespo-Blanco et al., 2022) implies that

$$||v||_{\mathcal{M}} \leq C|||\nabla v|||_{\mathcal{M}} \quad \forall v \in W_0^{1,\mathcal{M}}(\Omega).$$

Thus, on W $(:= W_0^{1,\mathcal{M}}(\Omega))$, we will work with an equivalent norm

$$||v|| := ||\nabla v||_{\mathcal{M}} \; (= |||\nabla v|||_{\mathcal{M}}).$$

The next critical embedding result was proved in (Ho & Winkert, 2023b).

Proposition 3

Let $\mathcal{G}$ be given in (7) and assume that $b_1(\cdot) > 0, b_2(\cdot) \geq 0, b_1, b_2 \in L^{\infty}(\Omega)$, and $s_1, s_2 \in C_+(\Omega)$. Then it holds that

(i). $W^{1,\mathcal{M}}(\Omega) \hookrightarrow L^{\mathcal{G}}(\Omega)$ *with* $s_1(x) \leq p^*(x)$ *and* $s_2(x) \leq q^*(x)$ *for all* $x \in \overline{\Omega}$.

(ii). *Moreover, if* $s_1(x) < p^*(x)$ *and* $s_2(x) < q^*(x)$ *then* $W^{1,\mathcal{M}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathcal{G}}(\Omega)$.

Due to ***Proposition* 3**, it follows that

$$S := \inf_{\varphi \in W \setminus \{0\}} \frac{||\varphi||}{||\varphi||_{\mathcal{G}}} > 0. \tag{10}$$

In the sequel, the notations $u_n \xrightarrow{s} u$ $(resp., u_n \xrightarrow{w} u,\ u_n \xrightarrow{w*} u)$ $in\ X\ as\ n \to \infty$,where X is some appropriate space, stand for that u_n converges strongly (resp., weakly, weakly*) to u in X as $n \to \infty$. The next result is called Lions-type concentration compactness principle stated for the solution space W, which is crucial to handle the lack of compactness issue. We refer to (Ha & Ho, 2024) for more details.

Proposition 4 (Ha & Ho, 2024)

Let $(\mathcal{A}_1) - (\mathcal{A}_2)$ hold and $\{u_n\}_{n \in N}$ is a "bounded sequence in W". We denote $\mathcal{R}(\overline{\Omega})$ by the "space of finite Radon measures" on Ω and assume that

$$u_n \xrightarrow{w} u \text{ in } W, \qquad \mathcal{M}(\cdot, |\nabla u_n|) \xrightarrow{w*} \eta \text{ in } \mathcal{R}(\overline{\Omega}), \qquad \mathcal{G}(\cdot, |u_n|) \xrightarrow{w*} \mu \text{ in } \mathcal{R}(\overline{\Omega}),$$

Then there exist families of distinct points $\{x_i\}_{i \in I}$, of non-negative numbers $\{\mu_i\}_{i \in I}, \{\eta_i\}_{i \in I} \subset (0, \infty)$, where I is an at most countable, such that

$$\mu = \mathcal{G}(\cdot, |u|) + \sum_{i \in I} \mu_i \delta_{x_i} \tag{11}$$

$$\eta \geq \mathcal{M}(\cdot, |\nabla u|) + \sum_{i \in I} \eta_i \delta_{x_i} \tag{12}$$

$$S \min\left\{\mu_i^{\frac{1}{p^*(x_i)}}, \mu_i^{\frac{1}{q^*(x_i)}}\right\} \leq \max\left\{\eta_i^{\frac{1}{p(x_i)}}, \eta_i^{\frac{1}{q(x_i)}}\right\} \tag{13}$$

where δ_{x_i} is the "Dirac mass" at x_i and S is given in (10).

We prepare to give proofs for our main results in the next section.

PROOFS OF MAIN RESULTS

The existence of a nontrivial solution

Along this section, we let $\theta = 1$ and all hypotheses in Theorem 1 are valid. The idea of proof is employed from (Chung & Ho, 2022). We also use C_i with $i \in N$ to denote positive constants, which depend only on the initial data and values may change differently lines through lines.

Due to the facts that $q^+ < \sigma$ and K is continuous, non-decreasing on $[0, t_0)$, we choose t_* so that $0 < t_* < \min\{1, t_0\}$, and

$$K(t_*)q^+ < K_0\sigma. \tag{14}$$

Define a modification of K as

$$\widetilde{K}(t) = \begin{cases} K(t) \text{ if } t \in [0, t_*] \\ K(t_*) \text{ if } t > t_* \end{cases}.$$

Clearly, $\widetilde{K}: [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and satisfies

$$K_0 \leq \widetilde{K}(t) \leq K(t_*) \text{ for all } t \in [0,\infty) \tag{15}$$

consequently,

$$K_0 t \leq \widehat{K}(t) := \int_0^t \widetilde{K}(\tau)d\tau \leq K(t_*)t \quad \text{for } t \in [0,\infty). \tag{16}$$

For any $\lambda > 0$, we consider the modified functional $J_\lambda: W \to \mathbb{R}$ associated with problem *(1)* as

$$J_\lambda(u) = \widehat{K}\left(\int_\Omega \mathcal{T}(x,\nabla u)\mathrm{d}x\right) - \lambda \int_\Omega F(x,u)\mathrm{d}x - \int_\Omega \widehat{G}(x,u)\mathrm{d}x, \forall u \in W$$

where $\widehat{G}(x,\cdot)$ is the anti-derivative of $g(x,\cdot)$, that is

$$\widehat{G}(x,t) = \frac{b_1(x)}{s_2(x)}|t|^{s_1(x)} + \frac{b_2(x)}{s_2(x)}\mu(x)^{\frac{s_2(x)}{q(x)}}|t|^{s_2(x)} \text{ for } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Using ***Proposition 2***, ***Proposition 3*** with standard arguments, the functional J_λ belongs to class C^1 with derivative $J'_\lambda: W \to W^*$ given by

$$\langle J'_\lambda(u),\varphi\rangle = \widetilde{K}\left(\int_\Omega \mathcal{T}(x,\nabla u)dx\right)\int_\Omega T(x,\nabla u).\nabla\varphi \,\mathrm{d}x - \lambda\int_\Omega f(x,u)\varphi \,\mathrm{d}x - \int_\Omega g(x,u)\varphi \,\mathrm{d}x$$

for all $u, \varphi \in W$.

We will look for a critical point u_λ of J_λ satisfying $\int_\Omega \mathcal{T}(x,\nabla u_\lambda)dx < t_*$, which becomes a solution of problem *(1)*.

The following lemma is a compactness result of J_λ. Recalling that a sequence $\{u_n\}_{n\in N} \subset W$ is said to be a (PS) sequence for J_λ in W at a level $L \in \mathbb{R}$ if it satisfies

$$J_\lambda(u_n) \to L \text{ and } J'_\lambda(u_n) \to 0 \text{ as } n \to \infty.$$

Lemma 1

Suppose that $\{u_n\}_{n\in N}$ is a (PS) sequence for J_λ in W at level $L \in \mathbb{R}$ and L satisfies

$$L < \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right) min\{K_0^{\tau_1}, K_0^{\tau_2}\} \min\{S^{n_1}, S^{n_2}\} \tag{17}$$

where $\tau_1 := \left(\frac{p^*}{q^*-p}\right)^-$, $\tau_2 := \left(\frac{q^*}{p^*-q}\right)^+$, $n_1 := \left(\frac{pq^*}{q^*-p}\right)^-$, $n_2 := \left(\frac{qp^*}{p^*-q}\right)^+$. Then, the sequence $\{u_n\}_{n\in N}$ admits a subsequence converging strongly in W.

Proof.

Let us claim firstly that $\{u_n\}_{n\in N}$ is bounded in W. One has from (15)-(16) that

$$\frac{K_0}{q^+}\int_\Omega \mathcal{M}(x,|\nabla v|)\,dx \leq \widehat{K}\left(\int_\Omega \mathcal{T}(x,\nabla v)dx\right) \leq \frac{K(t_*)}{p^-}\int_\Omega \mathcal{M}(x,|\nabla v|)\,dx, \forall\, v \in W. \tag{18}$$

Using ***Proposition* 1** together with (f_1), (14) and the above inequality, for $n \in N$ large it holds that

$$\begin{aligned} L + 1 + ||u_n|| &\geq J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n),u_n\rangle \\ &\geq \left(\frac{K_0}{q^+} - \frac{K(t_*)}{\sigma}\right)\int_\Omega \mathcal{M}(x,|\nabla u_n|)dx + \frac{\lambda}{\sigma}\int_\Omega [f(x,u_n)u_n - \sigma F(x,u_n)]dx \\ &\quad + \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right)\int_\Omega \mathcal{G}(x,|u_n|)dx \\ &\geq C_1(||u_n||^{p^-} - 1) \end{aligned} \tag{19}$$

which yields the boundedness of $\{u_n\}$ in W (due to $p^- > 1$). Thus, by means of a subsequence, it follows from ***Proposition 4*** that

$$u_n(x) \to u(x) \quad \text{a.a } x \in \Omega \tag{20}$$

$$u_n \overset{w}{\to} u \quad in \ \ W \tag{21}$$

$$\mathcal{M}(\cdot, |\nabla u_n|) \overset{w*}{\to} \eta \geq \mathcal{M}(\cdot, |\nabla u|) + \sum_{i \in I} \eta_i \, \delta_{x_i} \ \ in \ \ \mathcal{R}(\overline{\Omega}) \tag{22}$$

$$\mathcal{G}(\cdot, |u_n|) \overset{w*}{\to} \mu = \mathcal{G}(\cdot, |u|) + \sum_{i \in I} \mu_i \delta_{x_i} \ \ in \ \ \mathcal{R}(\overline{\Omega}) \tag{23}$$

$$S min\left\{\mu_i^{\frac{1}{p^*(x_i)}}, \mu_i^{\frac{1}{q^*(x_i)}}\right\} \leq max\left\{\eta_i^{\frac{1}{p(x_i)}}, \eta_i^{\frac{1}{q(x_i)}}\right\} \tag{24}$$

To obtain the strong convergence of a subsequence $\{u_n\}$ in W, we are going to show "$I = \emptyset$". Assuming for contradiction, there exists some $i \in I$. Using (20)-(24) and applying identical arguments used in the proof of Lemma 5.1 (Ha & Ho, 2024) , we obtain

$$\mu_i \geq min\{K_0^{\tau_1}, K_0^{\tau_2}\} \min\{S^{n_1}, S^{n_2}\}. \tag{25}$$

Meanwhile, the assumption (f_1), (18) and the property $(PS)_c$ of $\{u_n\}$ imply that

$$\begin{aligned} L + o_n(1) &\geq J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n), u_n\rangle \\ &\geq \left(\frac{K_0}{q^+} - \frac{K(t_*)}{\sigma}\right)\int_\Omega \mathcal{M}(x, |\nabla u_n|)dx + \frac{\lambda}{\sigma}\int_\Omega [f(x, u_n)u_n - \sigma F(x, u_n)]dx \\ &+ \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right)\int_\Omega \mathcal{G}(x, |u_n|)dx \end{aligned}$$

which leads to $L + o_n(1) \geq \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right)\int_\Omega \mathcal{G}(x, |u_n|)dx$.

Letting $n \to \infty$ in the last estimate and taking (23), (25) into account, we deduce

$$L \geq \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right)\eta_i \geq \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right) min\{K_0^{\tau_1}, K_0^{\tau_2}\} \min\{S^{n_1}, S^{n_2}\},$$

which contradicts to (17). Hence, we must have $I = \emptyset$. Thus, we deduce from (23) that

$$\int_\Omega \mathcal{G}(x, |u_n|)dx \to \int_\Omega \mathcal{G}(x, |u|)dx \ \ as \ n \to \infty.$$

Combining this with (20) and Lemma 4.3 (Brezis-Lieb type) in (Ha & Ho, 2024), we derive

$$u_n \overset{s}{\to} u \text{ in } L^{\mathcal{G}}(\Omega)$$

Finally, by virtue of Theorem 3.3 (Crespo-Blanco et al., 2022) and reasoning as in the last part of Theorem 5.1 (Ha & Ho, 2024), we obtain a strong convergent subsequence of $\{u_n\}$ in W. ■

The next lemma is about the geometry mountain pass property of J_λ.

Lemma 2

For any $\lambda > 0$, J_λ satisfies the following two facts.

(i). There are positive numbers δ and ρ such that $J_\lambda(u) \geq \rho$ for all $u \in W$ with $||u|| = \delta$.

(ii). There is a $\varphi_\lambda \in W$ with $||\varphi_\lambda|| > \delta$ holding $J_\lambda(\varphi_\lambda) < 0$.

Proof.

We show *(i)*. Thanks to ***Proposition 2*** and ***Proposition 3***, let us choose a constant $C_1 > 1$ such that

$$\max\{||u||_{m(\cdot)}, ||u||_{\mathcal{G}}\} \leq C_1||u|| \text{ for all } u \in W.$$

Assumptions (f_0) and (f_2) imply that

$$|F(x,t)| \le \frac{K_0\kappa}{2q^+\lambda}|t|^{p(x)} + C_2|t|^{m(x)} \text{ for all } t \in \mathbb{R} \text{ and a.a } x \in \overline{\Omega}.$$

where κ is given in (P). (27)

Combining ***Proposition 1*** with (8), one has

$$J_\lambda(u) \ge \frac{K_0}{2q^+}\int_\Omega \mathcal{M}(x,|\nabla u|)dx + \frac{K_0}{2q^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{K_0\kappa}{2q^+}\int_\Omega |u|^{p(x)}dx - C_2\int_\Omega |u|^{m(x)}dx - \frac{1}{s_1^-}\int_\Omega \mathcal{G}(x,|u|)dx$$
$$\ge \frac{K_0}{2q^+}\int_\Omega \mathcal{M}(x,|\nabla u|)dx - C_2\, max\left\{||u||_{m(\cdot)}^{m^-}, ||u||_{m(\cdot)}^{m^+}\right\} - \frac{1}{s_1^-} max\left\{||u||_{\mathcal{G}}^{s_1^-}, ||u||_{\mathcal{G}}^{s_2^+}\right\}.$$

for all $u \in W$. Thanks to (26)-(27), the last estimate implies

$$J_\lambda(u) \ge C_3\int_\Omega \mathcal{M}(x,|\nabla u|)dx - C_3\, max\{||u||^{m^-}, ||u||^{m^+}\} - C_3\, max\{||u||^{s_1^-}, ||u||^{s_2^+}\},$$

$\forall u \in W$.

Therefore, for any $u \in W$ with $||u|| = \delta \in (0,1)$, we get from ***Proposition 1*** and the last inequality that

$$J_\lambda(u) \ge C_3||u||^{q^+} - C_3||u||^{m^-} - C_3||u||^{s_1^-} = C_3\delta^{q^+} - C_3\delta^{m^-} - C_3\delta^{s_1^-}.$$

Since $q^+ < m^- < s_1^-$ (by (f_0)), for $\delta \in (0,1)$ small enough we deduce

$$\rho := C_3\delta^{q^+} - C_3\delta^{m^-} - C_3\delta^{s_1^-} > 0.$$

Now we show (ii). Let $u_* \in C_c^\infty(\Omega) \setminus \{0\}$ be fixed. Note that $F(x,t) > 0$ due to (f_1), then for any $t > 1$ we have

$$J_\lambda(tu_*) \le A_0 t^{q^+} - A_1 t^{s_1^-}$$

where $A_0 := \frac{K(t_*)}{p^-}\left(\int_\Omega \mathcal{M}(x,|\nabla u_*|)dx\right)$ and $A_1 := \frac{1}{s_2^+}\int_\Omega \mathcal{G}(x,|u_*|)dx$.

Since $q^+ < s_1^-$, the last inequality implies that there exists $t_1(\lambda) > 0$ sufficiently large such that $t_1(\lambda)||u_*|| > \delta$ and $J_\lambda(\varphi_\lambda) < 0$ *with* $\varphi_\lambda := t_1(\lambda)u_*$. Hence, (ii) follows. ∎

For each $\lambda > 0$, with φ_λ given in ***Lemma 2***, we define

$$c_\lambda := \inf_{h\in\Gamma_\lambda} \max_{t\in[0,1]} J_\lambda(h(t)). \tag{28}$$

where $\Gamma_\lambda := \{h \in C([0,1], W): h(0) = 0, h(1) = \varphi_\lambda\}$.

The next lemma involves the existence of a (PS) sequence for J_λ, which is crucial to get a critical point.

Lemma 3

It holds that $c_\lambda > 0$. Moreover, there exists a sequence $\{u_n\}_{n\in N} \subset W$ so that $\{u_n\}_{n\in N}$ is a (PS) sequence at level c_λ for J_λ in W, that is

$$J_\lambda(u_n) \to c_\lambda, \text{and } J'_\lambda(u_n) \to 0 \text{ as } n \to \infty.$$

The proof is followed by the combination of ***Lemma 2*** with a general version of "Mountain Pass Theorem" stated in Lemma 3.1 (Azorezo & Alonso, 1991). We omit the details for the brevity.

We are now in position proving Theorem 1.

Proof of Theorem 1

We take $\{u_n\}_{n\in N}$ as the sequence found in ***Lemma 3***. We are going to show that

$$c_\lambda \to 0 \text{ as } \lambda \to +\infty. \tag{29}$$

To this end, let $\{\lambda_n\}_{n\in N} \subset (0,\infty)$ be such that $\lambda_n \to \infty\ as\ n \to \infty$. Following the proof of ***Lemma 2***,for each $n \in N$ and $u_* \in C_c^\infty(\Omega)$ be fixed with $||u_*|| \neq 0$, $J_{\lambda_n}(tu_*)$ attains maximum at some $t = t_n >$ 0. Consequently, we have $\frac{d}{dt}J_{\lambda_n}(tu_*)\Big|_{t=t_n} = 0$; thus, it holds that

$$\widetilde{K}\left(\int_\Omega \mathcal{T}(x,t_n\nabla u_*)dx\right)\int_\Omega \mathcal{M}(x,|t_n\nabla u_*|)\,dx = \lambda_n \int_\Omega f(x,t_nu_*)t_nu_*\,dx + \int_\Omega \mathcal{G}(x,t_n|u_*|)\,dx. \tag{30}$$

Since $f(x,\tau)\tau > 0$ for a.a $x \in \Omega$ and $\tau \in \mathbb{R}$ (due to (f_1)), we obtain from (30) that

$$\widetilde{K}\left(\int_\Omega \mathcal{T}(x,t_n\nabla u_*)dx\right)\int_\Omega \mathcal{M}(x,|t_n\nabla u_*|)\,dx \geq \int_\Omega \mathcal{G}(x,t_n|u_*|)dx \geq\ min\left\{t_n^{s_1^-},t_n^{s_2^+},\right\}\int_\Omega \mathcal{G}(x,|u_*|)dx. \tag{31}$$

On the other hand, thanks to (15) we deduce

$$\widetilde{K}\left(\int_\Omega \mathcal{T}(x,t_n\nabla u_*)dx\right)\int_\Omega \mathcal{M}(x,|t_n\nabla u_*|)\,dx \leq K(t_*)\max\left\{t_n^{p^-},t_n^{q^+}\right\}\int_\Omega \mathcal{M}(x,|\nabla u_*|)\,dx. \tag{32}$$

Since $s_1^- > q^+$, (31) and (32) imply that $\{t_n\}_{n\in N}$ is bounded, hence, by extracting subsequence, if necessary, we have

$$t_{\rm n} \to \bar{t} \geq 0.$$

We are going to show that $\bar{t} = 0$. By contrast, we assume $\bar{t} > 0$. In (30), we pass the limit as $n \to \infty$. For the left side of (30), we have

$$\widetilde{K}\left(\int_\Omega \mathcal{T}(x,t_n\nabla u_*)dx\right)\int_\Omega \mathcal{M}(x,|t_n\nabla u_*|)\,dx \to \widetilde{K}\left(\int_\Omega \mathcal{T}(x,\bar{t}\nabla u_*)dx\right)\int_\Omega \mathcal{M}(x,|\bar{t}\nabla u_*|)\,dx < \infty$$

meanwhile, for the right side, since $\lambda_n \to \infty$ as $n \to \infty$, it holds

$$\lambda_n \int_\Omega f(x,t_nu_*)t_nu_*\,dx + \int_\Omega \mathcal{G}(x,t_n|u_*|)\,dx \to \infty.$$

Thus, we get the contradiction and so $\lim_{n\to\infty} t_n = 0$.

For each $n \in N$, we consider the path $\vartheta_n(t) := t\varphi_{\lambda_n}$, where φ_{λ_n} is given in ***Lemma 2***. Hence, one has $\vartheta_n \in \Gamma_{\lambda_n}$. In view of ***Lemma 2***-(ii), we have

$$\max_{t\geq 0} J_{\lambda_n}(tu_*) = \max_{t\in[0,t_1(\lambda)]} J_{\lambda_n}(tu_*) = \max_{t\in[0,1]} J_{\lambda_n}(t\varphi_{\lambda_n}) = \max_{t\in[0,1]} J_{\lambda_n}(\vartheta_n(t)).$$

Thus, formula (28) of c_λ provides

$$0 < c_{\lambda_n} \leq \max_{t\in[0,1]} J_{\lambda_n}(\vartheta_n(t)) = \max_{t\geq 0} J_{\lambda_n}(tu_*) = J_{\lambda_n}(t_nu_*) \leq \widehat{K}\left(\int_\Omega \mathcal{T}(x,t_n\nabla u_*)dx\right) \leq \frac{K(t_*)}{p^-}\max\left\{t_n^{p^-},t_n^{q^+}\right\}.$$

Because $\lim_{n\to\infty} t_n = 0$, we get (29). This result enables us to take a $\lambda^\# > 0$ such that

$$c_\lambda < \left(\frac{1}{\sigma} - \frac{1}{s_1^-}\right) min\{K_0^{\tau_1},K_0^{\tau_2}\}\,min\{S^{n_1},S^{n_2}\}, \qquad \forall\lambda > \lambda^\#.$$

Then, extracting a subsequence, if necessary, we deduce from ***Lemma 1***

$$u_n \xrightarrow{s} u_\lambda\ in\ W\ as\ n \to \infty.$$

Hence, u_λ is a critical point of the modified functional J_λ. Since $J_\lambda(u_\lambda) = c_\lambda > 0$, u_λ is nontrivial.

We need to return u_λ to a solution of problem *(1)*, to this end, we show that $\int_\Omega \mathcal{T}(x,\nabla u_\lambda)dx < t_*$. Using (f_1), one has

$$c_\lambda = J_\lambda(u_\lambda) - \frac{1}{\sigma}\langle J_\lambda'(u_\lambda), u_\lambda\rangle \geq \left(\frac{K_0}{q^+} - \frac{K(t_*)}{\sigma}\right)\int_\Omega \mathcal{M}(x, |\nabla u_\lambda|)dx.$$

Therefore, (29) follows

$$\lim_{\lambda\to\infty}\int_\Omega \mathcal{M}(x, |\nabla u_\lambda|)\,dx = 0, \tag{33}$$

leading to

$$\lim_{\lambda\to\infty}\int_\Omega \mathcal{T}(x, \nabla u_\lambda)dx = 0.$$

As a consequence, there is $\lambda^* > \lambda^\#$ so that $\int_\Omega \mathcal{T}(x, \nabla u_\lambda)dx \leq t_*, \forall\lambda > \lambda^*$. Finally, we show (9). Indeed, this conclusion can be obtained easily from (33) and the following fact

$$\min\{||u_\lambda||^{q^+}, ||u_\lambda||^{p^-}\} \leq \int_\Omega \mathcal{M}(x, |\nabla u_\lambda|)\,dx. \qquad \blacksquare$$

The multiplicity result

The proof of **Theorem 2** is almost similar with that in Theorem 1.2 (Ho & Sim, 2023). The key difference is that we use ***Proposition 4*** instead of Theorem 2.7 (Ho & Sim, 2023) to obtain the compactness of the energy functional. For the brevity, the proof details are not included here.

LIST OF ABBREVIATIONS USED

CCP: Concentration-compactness principle

PS: Palais-Smale

CONFLICTS OF INTEREST

"The authors declare that there is no conflict of interest regarding the publication of this paper."

AUTHOR CONTRIBUTION

"All authors contributed to the discussion and writing of the final version equally".

ACKNOWLEDGMENTS

"This research is funded by Ho Chi Minh City University of Technology under grant number T-KHUD-2024-05. We acknowledge Ho Chi Minh City University of Technology (HCMUT), VNU-HCM for supporting this study."

REFERENCES

1. Azorezo, J. G., & Alonso, I. P. (1991). Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans.Amer.Math.Soc*, *333*(2), 877–895. https://doi.org/10.1007/s13163-022-00453-y
2. Bahrouni, A., Rădulescu, V. D., & Repovš, D. D. (2019). Double phase transonic flow problems with variable growth: Nonlinear patterns and stationary waves. *Nonlinearity*, *32*(7), 2481–2495. https://doi.org/10.1088/1361-6544/ab0b03
3. Baroni, P., Colombo, M., & Mingione, G. (2018). Regularity for general functionals with double phase. *Calculus of Variations and Partial Differential Equations*, *57*(2), 62. https://doi.org/10.1007/s00526-018-1332-z
4. Benci, V., D'Avenia, P., Fortunato, D., & Pisani, L. (2000). Solitons in Several Space Dimensions: Derrick's Problem and Infinitely Many Solutions. *Archive for Rational Mechanics and Analysis*, *154*(4), 297–324. https://doi.org/10.1007/s002050000101
5. Bouaam, H., El Ouaarabi, M., & Melliani, S. (2025). Kirchhoff-type double-phase problems with variable exponents and logarithmic nonlinearity in Musielak-Orlicz Sobolev spaces. *Rendiconti Del Circolo Matematico Di Palermo Series 2*, *74*(5), 164. https://doi.org/10.1007/s12215-025-01283-9

6. Byun, S.-S., & Oh, J. (2020). Regularity results for generalized double phase functionals. *Analysis & PDE*, *13*(5), 1269–1300. https://doi.org/10.2140/apde.2020.13.1269

7. Cen, J., Vetro, C., & Zeng, S. (2023). A multiplicity theorem for double phase degenerate Kirchhoff problems. *Applied Mathematics Letters*, *146*, 108803. https://doi.org/10.1016/j.aml.2023.108803

8. Cherfils, L., & Il'yasov, Y. (2005). On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian. *Communications on Pure & Applied Analysis*, *4*(1), 9–22. https://doi.org/10.3934/cpaa.2005.4.9

9. Chung, N. T., & Ho, K. (2022). On a *p* ·-biharmonic problem of Kirchhoff type involving critical growth. *Applicable Analysis*, *101*(16), 5700–5726. https://doi.org/10.1080/00036811.2021.1903445

10. Colasuonno, F., & Perera, K. (2025). Critical growth double phase problems: The local case and a Kirchhoff type case. *Journal of Differential Equations*, *422*, 426–488. https://doi.org/10.1016/j.jde.2024.12.027

11. Colasuonno, F., & Squassina, M. (2016). Eigenvalues for double phase variational integrals. *Annali Di Matematica Pura Ed Applicata*, *195*(6), 1917–1959. https://doi.org/10.1007/s10231-015-0542-7

12. Crespo-Blanco, Á., Gasiński, L., Harjulehto, P., & Winkert, P. (2022). A new class of double phase variable exponent problems: Existence and uniqueness. *Journal of Differential Equations*, *323*, 182–228. https://doi.org/10.1016/j.jde.2022.03.029

13. Diening, L., Harjulehto, P., Hästö, P., & Ruzicka, M. (2011). *Lebesgue and Sobolev Spaces with Variable Exponents* (Vol. 2017). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-18363-8

14. El Ahmadi, M., Lamaizi, A., & Bouabdallah, M. (2024). On the superlinear Kirchhoff problem involving the double phase operator with variable exponents. *Journal of Elliptic and Parabolic Equations*, *10*(2), 1039–1061. https://doi.org/10.1007/s41808-024-00289-1

15. Fan, X., Zhang, Q., & Zhao, D. (2005). Eigenvalues of p(x)-Laplacian Dirichlet problem. *Journal of Mathematical Analysis and Applications*, *302*(2), 306–317. https://doi.org/10.1016/j.jmaa.2003.11.020

16. Farkas, C., Fiscella, A., & Winkert, P. (2021). Singular Finsler Double Phase Problems with Nonlinear Boundary Condition. *Advanced Nonlinear Studies*, *21*(4), 809–825. https://doi.org/10.1515/ans-2021-2143

17. Fiscella, A., Marino, G., Pinamonti, A., & Verzellesi, S. (2024). Multiple solutions for nonlinear boundary value problems of Kirchhoff type on a double phase setting. *Revista Matemática Complutense*, *37*(1), 205–236. https://doi.org/10.1007/s13163-022-00453-y

18. Fiscella, A., & Pinamonti, A. (2023). Existence and Multiplicity Results for Kirchhoff-Type Problems on a Double-Phase Setting. *Mediterranean Journal of Mathematics*, *20*(1), 33. https://doi.org/10.1007/s00009-022-02245-6

19. Ha, H. H., & Ho, K. (2024). Multiplicity results for double phase problems involving a new type of critical growth. *Journal of Mathematical Analysis and Applications*, *530*(1), 127659. https://doi.org/10.1016/j.jmaa.2023.127659

20. Ho, K., & Sim, I. (2016). On degenerate p (x) -Laplace equations involving critical growth with two parameters. *Nonlinear Analysis*, *132*, 95–114. https://doi.org/10.1016/j.na.2015.11.003

21. Ho, K., & Sim, I. (2023). On sufficient "local" conditions for existence results to generalized p (.)-Laplace equations involving critical growth. *Advances in Nonlinear Analysis*, *12*(1), 182–209. https://doi.org/10.1515/anona-2022-0269

22. Ho, K., & Winkert, P. (2023a). Infinitely many solutions to Kirchhoff double phase problems with variable exponents. *Applied Mathematics Letters*, *145*, 108783. https://doi.org/10.1016/j.aml.2023.108783

23. Ho, K., & Winkert, P. (2023b). New embedding results for double phase problems with variable exponents and a priori bounds for corresponding generalized double phase problems. *Calculus of Variations and Partial Differential Equations*, *62*(8), 227. https://doi.org/10.1007/s00526-023-02566-8

24. Liu, W., & Dai, G. (2018a). Existence and multiplicity results for double phase problem. *Journal of Differential Equations*, *265*(9), 4311–4334. https://doi.org/10.1016/j.jde.2018.06.006

25. Liu, W., & Dai, G. (2018b). Three ground state solutions for double phase problem. *Journal of Mathematical Physics*, *59*(12), 121503. https://doi.org/10.1063/1.5055300

26. Marcellini, P. (1989). Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Archive for Rational Mechanics and Analysis*, *105*(3), 267–284.

27. Perera, K., & Squassina, M. (2018). Existence results for double-phase problems via Morse theory. *Communications in Contemporary Mathematics*, *20*(02), 1750023. https://doi.org/10.1142/S0219199717500237

28. Zeng, S., Rădulescu, V. D., & Winkert, P. (2022). Double phase obstacle problems with variable exponent. *Advances in Differential Equations*, *27*(9/10). https://doi.org/10.57262/ade027-0910-611

29. Zhikov, V. V. (1987). Averaging of functionals of the calculus of variations and elasticity theory. *Mathematics of the USSR-Izvestiya*, *29*(1), 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958

30. Zuo, J., & Massar, M. (2024). On a double phase problem of Kirchhoff-type with variable exponent. *Discrete and Continuous Dynamical Systems - S*, *17*(12), 3368–3383. https://doi.org/10.3934/dcdss.2024039

VNUHCM Journal of Engineering and Technology 2026, 9(3): 3312-3325

Open Access Full Text Article

Bài báo nghiên cứu

MỘT SỐ KẾT QUẢ TỒN TẠI NGHIỆM CHO BÀI TOÁN KIRCHHOFF PHA KÉP VỚI CÁC PHI TUYẾN CHỨA TĂNG TRƯỞNG TRÊN TUYẾN TÍNH VÀ TỚI HẠN

Hoang Hai Ha[1,2,*], Doan Thi Thanh Xuan [3]

Use your smartphone to scan this QR code and download this article

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu một lớp phương trình phi tuyến, thuộc lĩnh vực phương trình đạo hàm riêng, chứa toán tử pha kép số mũ biến. Cụ thể hơn, chúng tôi xét các bài toán chứa toán tử có dạng

$$div\left(|\xi|^{p(x)-2}\xi + \mu(x)|\xi|^{q(x)-2}\xi\right)$$

cùng với số hạng phi địa phương. Các bài toán chứa toán tử dạng này xuất hiện trong các ứng dụng thuộc nhiều lĩnh vực khác nhau, chẳng hạn trong các quá trình phản ứng-khuếch tán, lý thuyết đàn hồi phi tuyến, cơ học chất lỏng hay vật lý lượng tử.

Một tính chất đặc biệt của toán tử ``pha kép'' là khả năng thay đổi liên tục giữa các pha khác nhau theo các điểm trong không gian. Hiện tượng này làm cho các bài toán liên quan toán tử pha kép trở nên khác biệt và chứa nhiều thách thức hơn so với các bài toán cổ điển Laplacian trước đó. Hơn thế nữa, bài toán chúng tôi xét trong nghiên cứu này cũng chứa các số hạng phản ứng (số hạng phi tuyến) với các tăng trưởng tổng quát, kết hợp giữa tăng trưởng ``dưới tới hạn'' và ``tới hạn'', do đó bao hàm các lớp phương trình rộng hơn so với một số nghiên cứu trước đó.

Mục tiêu của chúng tôi trong nghiên cứu này là thu được sự tồn tại nghiệm (yếu) của bài toán và dáng điệu nghiệm. Nghiệm bài toán được xây dựng trong không gian Musielak-Orlicz-Sobolev liên kết toán tử pha kép. Phương pháp nghiên cứu được xây dựng dựa trên các lý thuyết của giải tích biến phân, đưa nghiệm của bài toán về điểm tới hạn của phiếm hàm năng lượng tương ứng. Lý thuyết điểm tới hạn đòi hỏi tính compact của phiếm hàm năng lượng. Khi bậc tăng trưởng của các phi tuyến là dưới tới hạn, tính compact của phiếm hàm năng lượng thu được không khó khăn thông qua các phép nhúng Sobolev. Tuy nhiên, khi phi tuyến chứa bậc tăng trưởng tới hạn, tính compact của các phép nhúng Sobolev không còn nữa. Để khắc phục điều này, chúng tôi cần sử dụng kết quả về nguyên lý compact tập trung của Lions, được thiết lập cho các không gian với toán tử pha kép đã được công bố trước đó.

Từ khoá: toán tử pha kép, các bài toán Kirchhoff, số mũ biến, tăng trưởng tới hạn

[1]*Bộ môn Toán ứng dụng, Khoa Khoa học Ứng dụng, Trường Đại học Bách Khoa TPHCM, 268 Lý Thường Kiệt, phường Diên Hồng, Thành phố Hồ Chí Minh, Việt Nam*

[2]*Đại học Quốc Gia TPHCM, phường Linh Xuân, Thành phố Hồ Chí Minh, Việt Nam*

[3]*Đại học Công Nghiệp TPHCM, 12 đường Nguyễn Văn Bảo, phường Hạnh Thông, Thành phố Hồ Chí Minh, Việt Nam*

Liên hệ

Hoang Hai Ha, Bộ môn Toán ứng dụng, Khoa Khoa học Ứng dụng, Trường Đại học Bách Khoa TPHCM, 268 Lý Thường Kiệt, phường Diên Hồng, Thành phố Hồ Chí Minh, Việt Nam

Đại học Quốc Gia TPHCM, phường Linh Xuân, Thành phố Hồ Chí Minh, Việt Nam

Email: hoanghaiha@hcmut.edu.vn

Lịch sử

- Ngày nhận: 25-02-2026
- Ngày sửa đổi: 08-04-2026
- Ngày chấp nhận: 05-06-2026
- Ngày đăng: 11-09-2026

DOI : **https://doi.org/10.32508/vnuhcmj-et.v9i3.1577**

Trích dẫn bài báo này: Ha H H, Thanh Xuan D T. **MỘT SỐ KẾT QUẢ TỒN TẠI NGHIỆM CHO BÀI TOÁN KIRCHHOFFPHA KÉP VỚI CÁC PHI TUYẾN CHỨA TĂNG TRƯỞNG TRÊN TUYẾN TÍNH VÀ TỚI HẠN**. *VNUHCM J. Eng. Technol. 2026; 9(3): 3312-3325.*